1. Let C be a curve of genus 2 over a field k (which for simplicity may be assumed to be algebraically closed). Let O be a k-point of C, and let $\sim$ denote linear equivalence of divisors on C. Also let $C^{(2)}$ be the symmetric square of C, i.e. the set of unordered pairs of points of C, and denote pairs $\{P_1, P_2\}$ in $C^{(2)}$ by $P_1 + P_2$. Using the Riemann-Roch Theorem, show the following:

(a) If $P_1, P_2, Q_1, Q_2 \in C$ then there exist $R_1, R_2 \in C$ such that $P_1 + P_2 + Q_1 + Q_2 \sim R_1 + R_2 + 2O$ as divisors on C.

(b) In part (a), the element $R_1 + R_2 \in C^{(2)}$ is unique if and only if $P_1 + P_2 + Q_1 + Q_2$ is *not* linearly equivalent to $2O + K$, where K is the canonical divisor on C.

(c) Let $\equiv$ be the equivalence relation on $C^{(2)}$ given by taking $P_1 + P_2 \equiv Q_1 + Q_2$ if and only if either

(i) $P_1 + P_2 = Q_1 + Q_2$, or

(ii) $P_1 + P_2 \sim Q_1 + Q_2 \sim K$ as divisors on C.

Let $J = (C^{(2)}/\equiv)$, and denote the image of $\xi \in C^{(2)}$ in J by $[\xi]$. Then there is a well defined binary operation $\oplus$ making J an abelian group, given by the condition:
$[P_1+P_2]\oplus[Q_1+Q_2] = [R_1+R_2]$ if and only if $(P_1+P_2-2O)+(Q_1+Q_2-2O) \sim R_1+R_2-2O$.

Note: The points $P_1 + P_2 \in C^{(2)}$ such that $P_1 + P_2 \sim K$ form a projective line E in the surface $C^{(2)}$. Blowing down E, i.e. contracting E to a point, yields the surface $\mathrm{Jac}(C)$. Thus J in (c) above corresponds to the set of points of the Jacobian of C, and the group law on $\mathrm{Jac}(C)$ is as in (c).

2. Let k be a finite field, let K be a finite field extension of $k(x)$, and let $\pi : C \to \mathbb{P}^1_k$ be the corresponding branched cover of the projective x-line over k. Also, let $S = \pi^{-1}(\infty) \subset C$ (the points at infinity), let $r = \#S$, and let R be the ring of functions on the affine curve $C' = C - S$. Assume for simplicity that there is a k-point in S.

(a) Show that there is a natural surjective homomorphism $\mathrm{Div}(C) \to \mathrm{Div}(C')$, obtained by ignoring the points at ∞, and that this induces a surjective homomorphism $\alpha : \mathrm{Pic}^0(C) \to \mathrm{Pic}(C')$.

(b) Show that if $r = 1$ then α is an isomorphism.

(c) For general r, if $D \in \mathrm{Div}(C)$, let $[D]$ denote the image of D in $\mathrm{Pic}(C)$. Show that if $D \in \mathrm{Div}^0(C)$ and $[D] \in \ker(\alpha)$, then D is linearly equivalent (on C) to a divisor of degree 0 supported on S.

(d) Deduce that $\ker(\alpha)$ is a finite abelian group A having at most $r - 1$ generators, and thus $\mathrm{Pic}(C') \approx \mathrm{Pic}^0(C)/A$. Explain why this generalizes (b).

3. (a) Let $K = \mathbb{C}(x)$. Show that the field extension

$$K \subset K[y,z]/\big(y^2 - \pi x,\ z^3 - \frac{y+\sqrt{\pi}}{y-\sqrt{\pi}}\big)$$

is Galois with group S_3, and corresponds to a finite étale cover $Y \to \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$.

(b) Find a Galois finite étale cover $Z \to \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ with group S_3, together with an isomorphism $Z_{\mathbb{C}} := Z \times_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} Y$ which is compatible with the covering maps to $\mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$ and with the Galois actions of S_3.

4. (a) Let K be a field of characteristic p. Recall that every C_p-Galois field extension of K is of the form $L = K[y]/(y^p - y - a)$, for some $a \in K$, where the generator of C_p takes $y \mapsto y + 1$ (Artin-Schreier theorem).

(i) Show that $K[y]/(y^p - y - a)$ is a C_p-Galois field extension of K if and only if a is not of the form $u^p - u$, with $u \in K$.

(ii) Show that two such C_p-Galois extensions, L (as above) and $M = K[z]/(z^p - z - b)$ (where $b \in K$), are isomorphic if and only if there is an element $u \in K$ such that $u^p - u = b - a$. [Hint: $z = y + u$.]

(b) Let k be an algebraically closed field of characteristic p, and let $\alpha \in k$. Let Y_α be the curve $y^p - y = \alpha x$, and define $\pi_\alpha : Y_\alpha \to \mathbb{A}^1_k$ by $(x, y) \mapsto x$. Show that π is a C_p-Galois étale covering map, with Y_α irreducible for $\alpha \neq 0$.

(c) With k as in (b), let $\pi : Y \to \mathbb{A}^2_k$ be the cover of the (x, t)-plane given by $y^p - y = tx$. Show that π is étale and C_p-Galois. Explain why this cover can be regarded as a family of C_p-Galois étale covers Y_α of the x-line, parametrized by the points of the t-line over k. For which pairs α, β is Y_α isomorphic to Y_β as a Galois cover? Could a family with these properties exist in characteristic 0?

(d) Let Ω be the algebraic closure of $k(t)$. Find a G-Galois branched cover of $\mathbb{P}^1_\Omega$ that is *not* induced by any branched cover of $\mathbb{P}^1_k$. [Hint: Use part (c).] How does this differ from the situation in characteristic 0?