Math 702 Problem Set #1 Due Wed., Jan. 28, 2004

1. Let R be a Noetherian domain, and let $\mathfrak{p}$ be a non-zero prime ideal of R. Let $R_\mathfrak{p}$ be the local ring of R at $\mathfrak{p}$. Let $\hat{R}_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of R, viz. $\varprojlim R/\mathfrak{p}^n$.

(a) Show that if $\mathfrak{p}$ is maximal, then there is a natural inclusion $R_\mathfrak{p} \hookrightarrow \hat{R}_\mathfrak{p}$.

(b) Show that this conclusion does not necessarily hold for more general prime ideals $\mathfrak{p}$. [Hint: Try $R = k[x,y]$ and $\mathfrak{p} = (x)$. Is y a unit?]

2. (a) Let q be an odd prime power, and let $K_\infty = \mathbb{F}_q((t^{-1}))$, the infinite completion of $R = \mathbb{F}_q[t]$. Let $\mu = \{\text{roots of 1 in } R\}$. Verify the following:

(i) There is an element $s \in R$ whose norm is $N(s) = (\#\mu) + 1$, and having the following property: Every element of K_∞ can uniquely be written in the form $\alpha = \sum_{i=-\infty}^{n} a_i s^i$ for some integer n, with each $a_i \in \mu \cup \{0\}$ and $a_n \neq 0$. [Hint: There is a very simple choice for s.]

(ii) If α is as in (i), then α is a square in K_∞ iff a_n is a square in μ and n is even. [Hint: If n is even, is $s^{-n}\alpha$ a square?]

(iii) Let $S \subset K_\infty$ consist of the elements whose expressions involve only negative powers of s. Then S is a fundamental domain for the translation action of R on K_∞.

(b) Now let $K_\infty = \mathbb{R}$, the infinite completion of $R = \mathbb{Z}$. What is the analog of part (a)? [Hint: There are some differences, especially in (ii).] In the analog of (a)(iii), draw S.

(c) Redo (b) for $R = \mathbb{Z}[i]$ and for $R = \mathbb{Z}[\zeta_3]$, where ζ_3 is a primitive cube root of unity. [Note: The pictures of S should be a surprise.]

3. (a) Find all $\alpha \in \mathbb{Z}$ such that $\alpha^2 \equiv 2 \pmod{21}$. Then do the same for $\alpha \in \mathbb{Z}[i]$. (Hint: Chinese Remainder Theorem.)

(b) Find all $f(t) \in \mathbb{F}_{17}[t]$ such that $f(t)^2 \equiv t \pmod{t^2 - 1}$.

4. (a) Find the continued fraction expansion for $\sqrt{7}$.

(b) Find all the units in $\mathbb{Z}(\sqrt{7})$, the ring of integers of $\mathbb{Q}(\sqrt{7})$.

(c) What happens in parts (a) and (b) if $\sqrt{7}$ is replaced by $\sqrt{-7}$?

5. Let k be a field, and let $f(t) = t^4 + t^3 + t^2 + t$ and $g(t) = t^3 + t^2 + 1$ in $k[t]$.

(a) Find the g.c.d. of the polynomials $f(t)$ and $g(t)$ in $k[t]$.

(b) Find polynomials $X(t), Y(t) \in k[t]$ such that $f(t)X(t) + g(t)Y(t) = 1$.

(c) Find the continued fraction expansion for $f(t)/g(t)$ over $k[t]$.

6. Let $k = \mathbb{F}_3$, $R = k[t]$, and $K_\infty = k((t^{-1}))$.

(a) Show that $\sqrt{1+t^2} \in K_\infty$.

(b) Find the continued fraction expansion for $\sqrt{1+t^2}$ over R.

(c) Find all solutions $X(t), Y(t) \in R$ to the equation $X(t)^2 - (1+t^2)Y(t)^2 = 1$.

(d) Find all the units in the ring $k[t, \sqrt{1+t^2}]$, and interpret your answer in terms of functions on the affine curve $y^2 = 1 + x^2$ over k.

(e) What happens if $1 + t^2$ is replaced by $1 + t$, or by $1 - t^2$? What goes wrong, and why? What would analogous examples be with R, K_∞ replaced by $\mathbb{Z}, \mathbb{R}$?