Math 625 Problem Set #6 Due Monday, April 5, 2021

Read Hartshorne, Chapter III, Sections 5-9.

1. In Hartshorne, Chapter III, do problems 5.1, 7.1, 9.4. (In 9.4, you may assume that Y is a regular curve.)
Optional problems from Hartshorne: In Chapter III, problems 4.7, 5.3, 5.7, 9.3(a,b).

2. Suppose that $\pi : Y \to X$ is an étale morphism; i.e. it is locally given on the ring level by adjoining elements $y_1, \dots, y_m$ subject to relations $f_1, \dots, f_m$ such that the relative Jacobian matrix $(\partial f/\partial y)$ is invertible.

a) Show that if X is a smooth k-scheme then so is Y.

b) Let $\phi : Z \to X$ be any morphism, and consider the fibre product $Z \times_X Y = \{(z, y) \mid \phi(z) = \pi(y)\}$. Show that the first projection $p_1 : Z \times_X Y \to Z$ is étale. (Hint: First show that p_1 is defined by the same equations as define π.)

3. Suppose that $\pi : Y \to X$ is a finite étale cover of smooth connected k-schemes of dimension d. Let ϕ be an automorphism of the cover, and suppose that $\phi(Q_0) = Q_0$ for some $Q_0 \in Y$.

a) Show that $Y \times_X Y$ is a smooth scheme of dimension d, and that the first projection $p_1 : Y \times_X Y \to Y$ is étale. (Hint: problem 2.)

b) Define $s, t : Y \to Y \times_X Y$ by $s(Q) = (Q, Q)$ and $t(Q) = (Q, \phi(Q))$. Show that s and t are each sections of p_1, and that each is an isomorphism onto its image, with inverse equal to p_1.

c) Show that the images of s and of t are closed connected subvarieties of $Y \times_X Y$; are of dimension d; and intersect. Deduce that these images are equal.

d) Conclude that ϕ is the identity.

e) In the case of $k = \mathbb{C}$, why is the conclusion of (d) expected?

4. Let X and Y be smooth connected k-schemes, and let $\pi : Y \to X$ be a finite surjective morphism of degree d whose restriction to some Zariski dense open subset is étale. (We say that π is a *branched cover* of degree d.)

a) Show that the covering group has at most d elements. (Hint: First reduce to the étale case by restricting to an open set, and then use problem 3.)

b) For $d = 3$, give an example where the covering group has exactly three elements, and an example where it has fewer than three elements.

5. Let $n, i, m, q_1, \dots, q_m$ be integers, with $n, m > 0$ and $i \geq 0$. Let $\mathcal{F} = \bigoplus_{j=1}^{m} \mathcal{O}(q_j)$ on $\mathbb{P}^n_k$ for some algebraically closed field k. Find the dimension of $H^i(\mathbb{P}^n, \mathcal{F})$ explicitly in terms of the integers $n, i, q_1, \dots, q_m$.