Math 625 Problem Set #4 Due Monday, March 15, 2021

Read Hartshorne, Chapter IV, sections 5 and 6.

1. In Hartshorne, Chapter IV, do problems 4.7-4.9. Optional: problems 3.3, 5.1.

2. Let X be a smooth connected projective curve of genus $g \geq 2$ over an algebraically closed field k. Let $X^{(g)}$ be the gth-fold symmetric power of X. Identify effective divisors D of degree g on X with points of $X^{(g)}$.

a) Show that for all D in some dense open subset of $X^{(g)}$, the complete linear system $|D|$ has dimension 0. [Hint: What is the dimension of $|K|$? What is the dimension of $|K - P_1|$, if P_1 is not a base point of $|K|$? ... What about the dimension of $|K - D|$?]

b) Deduce that $X^{(g)} \to \mathrm{Pic}^0(X)$ is injective on a Zariski open dense subset. (Here the map is given by $D \mapsto [D - D_0]$ for some fixed effective divisor D_0 on X of degree g.)

3. Fix $X = \mathbb{P}^2$ over an algebraically closed base field k. Show that there is a fine moduli space for the lines in X, and that this moduli space is isomorphic to $\mathbb{P}^2$. (First, you have to describe the corresponding functor that you will represent.)

4. Let X be a smooth connected projective curve, and let $n \geq 3$. Show that the n-canonical map associated to the linear system $|nK|$ is an embedding if and only if the genus of X is greater than 1.

5. Let X be a smooth connected projective curve of genus g. For any positive integer n, say that X has property D_n if for every choice of distinct points $P_1, \ldots, P_n \in X$, there is a rational function on X with poles of order exactly 1 at each P_i, and no other poles. What can you say about the set of n's such that X satisfies property D_n?