Math 625 Problem Set #3 Due Monday, March 1, 2021

Read Hartshorne, Chapter IV, section 4.

1. In Hartshorne, Chapter IV, do problems 1.7, 3.2, 4.3.

2. Recall that if k is an algebraically closed field then a smooth connected projective k-curve X has genus 0 if and only if $X \approx \mathbb{P}^1_k$.

a) Show by example that this assertion is false if k is not assumed algebraically closed.

b) What goes wrong with the proof in that case? (Recall that Riemann-Roch still holds for non-algebraically closed fields.)

c) Show that over a general field k, if X has genus 0, then $X \approx \mathbb{P}^1_k$ iff X has a k-point.

3. Show that if k is algebraically closed of characteristic $p \neq 0$, then there are unramified covers $X \to \mathbb{A}^1_k$ such that the smooth projective model of X has arbitrarily large genus.

4. Redo PS2 problem 5(c) for a regular dodecahedron instead of a tetrahedron.

5. If $Y \to X$ is a Galois branched cover of regular connected schemes of dimension 1, with Galois group G, and if $Q \in Y$, recall that the *decomposition group* $D_Q \subset G$ consists of the elements $g \in G$ such that $g(Q) = Q$.

a) Show that D_Q is a subgroup of G.

b) Show that the order of D_Q is equal to the ramification index e_Q if $Y \to X$ is a branched cover of curves defined over an algebraically closed field k. Is this true for *all* fields k? Is an inequality true in one direction?

c) Let $k = \mathbb{C}$, let $Y = \mathbb{A}^1$ be the y-line, let $X = \mathbb{A}^1$ be the x-line, and consider the cover $Y \to X$ given by $y^2 = x$. Show that this cover is Galois; find the Galois group; and find the branch locus. Also find D_Q if $Q \in Y$ lies over $x = 0$. Redo this with $x = 1$ and with $x = -1$. What if instead $k = \mathbb{R}$?

d) Let $Z \subset \mathbb{A}^3$ be the affine curve given by $y^2 = x$, $z^2 = x - 1$, and let $\pi : Z \to X = \mathbb{A}^1$ be the cover given by projection onto the x-coordinate. Redo part (c) for this cover.

e) Let $X = \operatorname{Spec} \mathbb{Z}$, let $Y = \operatorname{Spec} \mathbb{Z}[i]$, and let $\pi : Y \to X$ be the morphism corresponding to the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$. Show that this cover is Galois, and find the Galois group and branch locus. Then find the decomposition group D_Q if $Q \in Y$ lies over the point corresponding to the prime (2). Redo this with the prime (3) and with the prime (5).