Math 625 Problem Set #2 Due Thursday, Feb. 18, 2021

Read Hartshorne, Chapter IV, section 3.

1. In Hartshorne, Chapter IV, do problems 1.5, 2.1, 2.5, 3.1.

2. Let k be an algebraically closed field of characteristic $p \neq 0$.

(a) Find the genus of the smooth completion Y of the affine curve Y° given by $y^p - y = x$. [Hint: Project onto the y-axis.]

(b) Let X be the projective x-line over k. Show that the projection map $Y^\circ \to \mathbb{A}^1_k$ onto the x-axis extends to a morphism $Y \to X$, and find where this morphism ramifies.

(c) By combining parts (a) and (b), explain how this example illustrates that the usual Hurwitz formula for tamely ramified covers $Y \to X$ of degree n, i.e. $2g_Y - 2 = n(2g_X - 2) + \sum_{Q \in X}(e_Q - 1)$, does not carry over to wildly ramified covers in characteristic $p > 0$.

3. Let k be a field.

a) Show that if k is algebraically closed and of characteristic 0, then $\mathbb{A}^1_k$ is simply connected (i.e. it has no non-trivial connected étale covers) .

b) Prove the converse of (a).

4. Let $\pi : Y \to X$ be a covering space map of Riemann surfaces (in the complex metric topology), where X is compact and of topological genus 1.

(a) Show that if π is of finite degree, then Y is also compact, and also of topological genus 1.

(b) Show that the above conclusion fails if π is of infinite degree. [Hint: Consider the universal cover.] Why doesn't this phenomenon arise in the algebraic situation (i.e. for morphisms)?

5. Let Y be a smooth connected curve over a field k. Let G be a finite group that acts faithfully on Y as a group of automorphisms. Suppose that the order of G is not divisible by the characteristic of k.

a) Show that there is a smooth curve X and a morphism $\pi : Y \to X$ which is a Galois branched cover having Galois (covering) group G, such that the fibres of π are the orbits of G. We call X the *quotient* Y/G. [Hint: First handle the affine case $Y = \operatorname{Spec} S$, by taking $X = \operatorname{Spec} S^G$, where S^G is the subring of G-invariants of S.]

b) Suppose that k is algebraically closed, and let $Q \in Y$. Show that Q is stabilized by some element of G other than the identity if and only if Q is a ramification point of $Y \to X$. Show that the number of elements of G that stabilize Q (including the identity) is equal to e_Q.

c) Let $k = \mathbb{C}$, and let $Y = \mathbb{P}^1_{\mathbb{C}}$, the Riemann sphere. Let G be the symmetry group of the solid tetrahedron T, where T is viewed as inscribed in the sphere. Consider the action of G on Y that is induced by the action of G on T. Describe the resulting cover $Y \to X = Y/G$ by giving its degree, the number of branch and ramification points, and the ramification indices; giving the genus of X; and checking that the Hurwitz formula holds.