Math 625 Problem Set #1 Due Monday, Feb. 8, 2021

Read Hartshorne, Chapter IV, sections 1,2.

1. In Hartshorne, Chapter IV, do problems 1.2, 1.3, 1.6.

2. Let k be an algebraically closed field of characteristic 0.

a) For $n \geq 1$, let C_n be the affine k-curve given by $y^2 = \prod_{j=1}^n (x-j)^j$. Find the genus of the smooth projective curve that is birationally isomorphic to C_n, as a function of n.

b) Let X be a smooth connected projective curve of genus g over k, and let $f : X \to \mathbb{P}^1$ be a non-constant morphism. Find an upper bound on g in terms of the degree of f and the number of branch points of f.

3. Let X be a smooth connected projective curve of genus 3 over an algebraically closed field k of characteristic 0. Let $P_0 \in X$.

a) Show that there is a rational function f on X with a pole of order exactly 6 at P_0, and having no other poles.

b) Find the degree of the morphism $X \to \mathbb{P}^1_k$ corresponding to f.

c) Show that there are (at least) three distinct constants $c_1, c_2, c_3 \in k$ such that for each i, the rational function $f - c_i$ has a multiple zero at some point $P_i \in X$. [Hint: Problem 2(b).]

4. Prove the analog of the Riemann-Hurwitz formula for covers $Y \to X$ of smooth complex connected projective curves (compact Riemann surfaces), with respect to the *topological* genus. Do this in two ways:

(a) Use compatible triangularizations of both Riemann surfaces, with the sets of vertices including all the branch points and ramification points. For both X and Y count

$$(\# \text{ of vertices}) - (\# \text{ of edges}) + (\# \text{ of faces})$$

and use that this alternating sum is equal to the topological Euler characteristic (which is $2-2g$ if the topological genus is g).

(b) Alternatively, find the topological Euler characteristic of a Riemann surface of topological genus g with r points removed. (For this, you may use that the topological Euler characteristic of a topological surface V is $\operatorname{rk} H^0(V,\mathbb{Z}) - \operatorname{rk} H^1(V,\mathbb{Z}) + \operatorname{rk} H^2(V,\mathbb{Z})$; that $\operatorname{rk} H^0(V,\mathbb{Z})$ is 1 if V is connected; that $H^1(V,\mathbb{Z})$ is the abelianization of $\pi_1(V)$; and that $\operatorname{rk} H^2(V,\mathbb{Z})$ is 1 if V is compact and orientable and is 0 otherwise.) Then use that for a covering space $Y^\circ \to X^\circ$ of degree n, the topological Euler characteristic of Y° is equal to n times that of X°.

5. Let Y be a smooth complex connected projective curve. Prove that the genus of Y as an algebraic curve is equal to the topological genus of Y as a Riemann surface. [Hint: First do this for the case of $Y = \mathbb{P}^1$. For more general Y, show that there is a non-constant morphism $Y \to \mathbb{P}^1$, of finite degree n. Then use the Riemann-Hurwitz formula for the algebraic version of genus, together with its topological analog (cf. Problem 4 above).]