1. Suppose $k \subset K$ is a separable field extension of degree n.

a) Show that $K \approx k[x]/(f(x))$ for some irreducible polynomial $f(x) \in k[x]$ of degree n, and that $K \otimes_k K \approx K[y]/(f(y))$ as K-algebras. [Hint: Identify each side of the second isomorphism with $k[x, y]/(f(x), f(y))$.]

b) Deduce that if K is Galois over k, then $f(y)$ splits over K, and $K \otimes_k K \approx K^n$ as K-algebras. [Hint: Use separability and the Chinese Remainder Theorem.]

c) Verify (b) explicitly in the case that $k = \mathbb{Q}$ and $K = \mathbb{Q}(i)$.

d) If K is not Galois over k, is it still necessarily true that $K \otimes_k K \approx K^n$?

2. Let R be an ordered field (i.e. a field with an ordering "$\le$" that satisfies the usual compatibilities with addition and multiplication) whose squares are the non-negative elements. Suppose that the elements of $R[x]$ satisfy the intermediate value theorem with respect to the ordering (viewing these polynomials as functions from R to R). Let $C = R[x]/(x^2 + 1)$.

a) Show that R has characteristic 0, and that every odd degree polynomial over R has a root in R. Deduce that every non-trivial Galois extension of R has even degree.

b) Show that C is a field that strictly contains R, that every element of C is a square of an element of C, and that C has no field extensions of degree 2. [Hint: Quadratic formula.]

c) Show that if $R \subset C \subseteq L$ are finite field extensions and L is Galois over R with group G, then G is a 2-group. [Hint: Let $H \subseteq G$ be a Sylow 2-subgroup, and let K be the fixed field of H.]

d) In the situation of (c), show that $L = C$. [Hint: If not, $\mathrm{Gal}(L/C)$ has a subgroup E of index 2; and considering the extension $C \subseteq L^E$ (= fixed field) yields a contradiction.]

e) Conclude that C is algebraically closed. [Hint: If $C \subset K$ is a non-trivial field extension, let L be the Galois closure of K over R, and apply (d).]

f) Deduce in particular that the field $\mathbb{C}$ of complex numbers is algebraically closed.

3. a) Let p be a prime number, and let G be a subgroup of S_p. Suppose that G contains a transposition and a p-cycle. Show that $G = S_p$.

b) Suppose that $f(x) \in K[x]$ is a separable irreducible polynomial of degree p (where p is prime), and let G be the Galois group of f over K. Show that G is a subgroup of S_p; that p divides the order of G; and that G contains a p-cycle. [Hint: What is $[K[x]/(f(x)) : K]$?]

c) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible of degree p and that exactly two of its roots do not lie in $\mathbb{R}$. Let G be the Galois group of f. Show that G contains a transposition, and deduce that G is isomorphic to S_p.

d) Deduce that $3x^5 - 6x - 2$ is not solvable by radicals.

4. a) Prove that any polynomial $f(x) \in \mathbb{Q}[x]$ of degree < 5 is solvable by radicals.

b) Find an $\alpha \in \bar{\mathbb{Q}}$ whose irreducible polynomial over $\mathbb{Q}$ has degree 5, and is solvable by radicals.

5. Let p be a prime number, and let $K \subset L$ be a field extension of degree p that is separable but not Galois. Let $\tilde{L}$ be the Galois closure of L over K. Show that $\tilde{L}$ does not contain *any* subfield M which is Galois over K of degree p. [Hint: Observe that $\mathrm{Gal}(\tilde{L}/K) \subseteq S_p$, and then consider the order of $\mathrm{Gal}(\tilde{L}/LM)$.]

6. For which positive integers n is it possible, with straightedge and compass, to divide *any* given angle into n equal parts? Prove your assertion.