Optional problem for Math 603, spring 2016, Problem Set 11

Hilbert proved the following Irreducibility Theorem:

Theorem. *Let $s_1, \ldots, s_m, x_1, \ldots, x_n$ be transcendentals over $\mathbb{Q}$, and consider an irreducible polynomial $f(s_1, \ldots, s_m, x_1, \ldots, x_n)$ over $\mathbb{Q}$. Then there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$ such that $f(\alpha_1, \ldots, \alpha_m, x_1, \ldots, x_n)$ is an irreducible polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$. Moreover, if a non-zero polynomial $g \in \mathbb{Q}[s_1, \ldots, s_m]$ is given in advance, then the α's can be chosen so that $g(\alpha_1, \ldots, \alpha_m) \neq 0$.*

a) Verify this explicitly in the case that $m = n = 1$, $f(s, x) = x^3 - s$, $g(s) = s^2 - 1$. Which values of α work?

b) Show that if $\mathbb{Q}(s_1, \ldots, s_m) \subseteq L$ is a finite field extension, then there is an irreducible polynomial $F \in \mathbb{Q}[s_1, \ldots, s_m, x]$ such that L is the fraction field of $\mathbb{Q}[s_1, \ldots, s_m, x]/(F)$.

c) Use Hilbert's theorem to show that there is an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that the extension $\mathbb{Q} \subseteq \mathbb{Q}[x]/(f)$ is Galois with group S_n. [Hint: Part (b) and Problem Set 10 #4.]

d) Still assuming Hilbert's theorem, conclude that the field K in problem 5 on Problem Set 11 can be chosen to be finite over $\mathbb{Q}$. [Hint: Reduce to the case of part (c) above.]