Math 603 Problem Set #11 Due Monday, April 18, 2016

1. a) Let $n > 1$, let $p > n$ be prime, and let K be a field of characteristic p. In the notation of problems 3 and 4 of PS10, let $\mathcal{K} = K(s_1, \ldots, s_n)$, let $\mathcal{L} = K(x_1, \ldots, x_n)$, and let $\mathcal{M} = \mathcal{L}[\sqrt[p]{x_1}]$. Show that $\mathcal{M}$ is not normal over $\mathcal{K} = K(s_1, \ldots, s_n)$.

b) In the situation of part (a), let $\mathcal{N}$ be the maximal purely inseparable extension of $\mathcal{K}$ contained in $\mathcal{M}$. Show that $\mathcal{N} = \mathcal{K}$. [Hint: If not, show that $[\mathcal{N} : \mathcal{K}] = p$, and deduce that $\mathcal{M} = \mathcal{L}\mathcal{N}$. Conclude that $\mathcal{M}$ is Galois over $\mathcal{N}$, and then use PS10 #7(b) to obtain a contradiction.]

c) Deduce that the conclusion of PS10 #5(c) does not necessarily hold if the given field extension is not normal.

2. Let p be a prime number. Recall Eisenstein's Irreducibility Criterion: If $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ where $p \mid a_i$ for all $i < n$, $p \not| a_n$, and $p^2 \not| a_0$, then f is irreducible over $\mathbb{Q}$. Use this to show that the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$, and hence is the minimal polynomial of ζ_p. [Hint: First set $y = x - 1$.]

3. For $n \geq 1$ let $K_n = \mathbb{Q}(\zeta_n)$.

a) Show that K_n is Galois over $\mathbb{Q}$, and describe $\mathrm{Gal}(K_n/\mathbb{Q})$ in terms of n. In particular, what is this Galois group when $n = 5$? 6? 7? 8? 12?

b) For which n is this extension abelian? of order 2? of order 3? For which n does it have a quotient of order 3?

c) Show that K_n is Galois over $K_n^+ := \mathbb{Q}(\zeta_n + \zeta_n^{-1})$, and that K_n^+ is Galois over $\mathbb{Q}$. Find $[K_7 : \mathbb{Q}]$, $[K_7 : K_7^+]$, and $[K_7^+ : \mathbb{Q}]$. Also find $\mathrm{Gal}(K_7/K_7^+)$ and $\mathrm{Gal}(K_7^+/\mathbb{Q})$.

d) Find a Galois extension of $\mathbb{Q}$ having degree 5. [Hint: See part (c).]

4. Find the Galois group of (the splitting field of) each of the following polynomials.

a) $x^3 - 10$ over $\mathbb{Q}$, and over $\mathbb{Q}(\sqrt{-3})$.

b) $x^4 - 5$ over $\mathbb{Q}$, and over $\mathbb{Q}(i)$.

c) $x^4 - t$ over $\mathbb{R}(t)$, and over $\mathbb{C}(t)$.

5. Show that for every finite group G, there are field extensions $\mathbb{Q} \subseteq K \subseteq L$ such that L is a finite Galois extension of K with $\mathrm{Gal}(L/K) = G$. (Remark: $[K : \mathbb{Q}]$ is allowed to be infinite.) [Hint: First show the result for $G = S_n$, using PS10 #4.]

6. Let $L = \mathbb{C}(x, y)$, $M = \mathbb{C}(x^2, xy, y^2) \subset L$, and $K = \mathbb{C}(x^2, y^2) \subset M$. Find $[L : M]$, $[M : K]$, $[L : K]$. Is L Galois over M? Is M Galois over K? Is L Galois over K? For those extensions that are Galois, find the Galois group.

7. Let K and L be finite extensions of a field k, and let KL be their compositum (inside some fixed algebraic closure).

a) Find a surjective k-algebra homomorphism $\pi : K \otimes_k L \twoheadrightarrow KL$.

b) Suppose that K is Galois over k. Show that π is an isomorphism if only if $K \cap L = k$. [Hint: Find $\dim_k(K \otimes_k L)$ and $\dim_k(KL)$.]

c) Does (b) still hold if K is no longer assumed Galois over k?