Math 603 Problem Set #10 Due Monday, April 11, 2016

1. Suppose that L is an algebraic extension of a field K that is purely inseparable. Let p be the characteristic of K. Let $F_0 = \mathbb{F}_p(t)$ and let $F_1 = F_0(t^{1/p})$. Show that there is an embedding $\phi : F_1 \hookrightarrow L$ such that $\phi^{-1}(K) = F_0$, so that there is a diagram as follows:

$$\begin{array}{ccc} F_1 & \hookrightarrow & L \\ | & & | \\ F_0 & \hookrightarrow & K \end{array}$$

2. Let $F = \mathbb{Z}/p\mathbb{Z}$, let $L = F(x, y)$, and let $K = F(x^p, y^p)$. Show that L is a finite field extension of K, but that there are infinitely many fields between K and L. Is $L = K[\alpha]$ for some $\alpha \in L$? Is L separable over K?

3. Let K be a field, and for $i = 1, \dots, n$ let $s_i \in K[x_1, \dots, x_n]$ be the i-th elementary symmetric polynomial in $x_1, \dots, x_n$.

a) Show that if K is algebraically closed and if $a_1, \dots, a_n \in K$ then there exist $b_1, \dots, b_n \in K$ such that $s_i(b_1, \dots, b_n) = a_i$ for all i. [Hint: Take the roots of the polynomial $Z^n - a_1 Z^{n-1} + a_2 Z^{n-2} - \cdots + (-1)^n a_n \in K[Z]$.]

b) Show that if $f(Y_1, \dots, Y_n)$ is any non-zero polynomial in $K[Y_1, \dots, Y_n]$, then the substitution $f(s_1, \dots, s_n)$ is not the zero polynomial in $K[x_1, \dots, x_n]$. [Hint: First reduce to the case that K is algebraically closed, by considering $\bar{K}$. Then use PS9 problem 8 and part (a) above.] This says that $s_1, \dots, s_n \in K[x_1, \dots, x_n]$ are algebraically independent.

4. a) In the notation of problem 3, show that $K(x_1, \dots, x_n)$ is the splitting field over $K(s_1, \dots, s_n)$ of the polynomial $Z^n - s_1 Z^{n-1} + s_2 Z^{n-2} - \cdots + (-1)^n s_n$.

b) Show that the extension $K(s_1, \dots, s_n) \subseteq K(x_1, \dots, x_n)$ is Galois.

c) Show that the Galois group G is the symmetric group S_n. [Hint: Show that $S_n \subseteq G$ and that $|G| = [K(x_1, \dots, x_n) : K(s_1, \dots, s_n)] \le n!$.]

5. Let L be a normal field extension of K, and let K_0 be the maximal purely inseparable extension of K contained in L. View L as contained in a fixed algebraic closure $\bar{K}$ of K.

a) Let $\beta \in L$, and let $\beta_1, \dots, \beta_n \in \bar{K}$ be the distinct images of β under the K-embeddings $L \hookrightarrow \bar{K}$ (listing each image only *once*, regardless of multiplicity). Let $f(x) = \prod(x - \beta_i)$. Show that $f(x) \in K_0[x]$. [Hint: Show $f(x) \in L[x]$, and then show that the coefficients of f are mapped to themselves under each K-embedding $L \hookrightarrow \bar{K}$.]

b) In part (a), show that $f(x)$ is the minimal polynomial of β over K_0. [Hint: Show that every β_i is a root of the minimal polynomial.]

c) Conclude that L is separable over K_0.

6. Let $K = \mathbb{F}_p(t)$ and let $L = K[\sqrt[2p]{t}]$, where p is an odd prime number.

a) Find the maximal separable extension K' of K in L, and the maximal purely inseparable extension K_0 of K in L.

b) Show *explicitly* in this example that L is the compositum of K' and K_0, by expressing $\sqrt[2p]{t}$ as a combination of elements from K' and K_0.

c) What if instead $p = 2$?

7. a) Show that every purely inseparable extension is normal.

b) More generally, show that if M is normal over N, and N is purely inseparable over K, then M is normal over K.