Math 603 Problem Set #9 Due Monday, April 4, 2016

1. Which of the following rings R are discrete valuation rings? For those that are, find the fraction field $K = \operatorname{frac} R$, the residue field $k = R/\mathfrak{m}$ (where $\mathfrak{m}$ is the maximal ideal), and a uniformizer π. For the others, explain why not (full proofs not required). $\mathbb{Z}$, $\mathbb{Z}_{(5)}$, $\mathbb{Z}[1/5]$, $\mathbb{R}[x]$, $\mathbb{R}[x]_{(x-2)}$, $\mathbb{R}[x, 1/(x-2)]$, $\mathbb{Q}[x]_{(x^2+1)}$, $\mathbb{C}[x,y]_{(x,y)}$, $\big(\mathbb{R}[x,y]/(x^2+y^2-1)\big)_{(x-1,y)}$, $\big(\mathbb{R}[x,y]/(y^2-x^3)\big)_{(x,y)}$.

2. a) Find the degree of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$, and also find its minimal polynomial.
b) Do the same for $\beta = \sqrt{3 + \sqrt[3]{2}}$.
c) Is $\mathbb{Q}(\alpha)$ normal over $\mathbb{Q}$? Is $\mathbb{Q}(\beta)$?

3. Let K be a field, and $f(x) \in K[x]$. Assume that K has characteristic 0. Let $n \geq 1$.
a) Let L be a finite field extension of K, and let $\alpha \in L$. Show that α is a root of f with multiplicity n if and only if $0 = f(\alpha) = f'(\alpha) = \cdots = f^{(n-1)}(\alpha) \neq f^{(n)}(\alpha)$.
b) Show that f has a root (in some extension of K) of multiplicity at least n if and only if $(f(x), f'(x), \ldots, f^{(n-1)}(x))$ is a proper ideal of $K[x]$.
c) What if instead K has non-zero characteristic?

4. For each of the following fields K, explicitly find the group $\operatorname{Aut} K$ of all automorphisms of K (as a field): $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt[3]{2}]$, $\mathbb{Q}[\zeta_7]$, $\mathbb{Q}[\zeta_3, \sqrt[3]{2}]$. (Here $\zeta_n = e^{2\pi i/n}$, a primitive nth root of unity.)

5. Let $K = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{2+\sqrt{2}}]$.
a) Find the multiplicative inverse of $\sqrt{2+\sqrt{2}}$ in L (as a polynomial in $\sqrt{2+\sqrt{2}}$).
b) Show $K \subset L$. What is $[K:\mathbb{Q}]$? $[L:K]$? $[L:\mathbb{Q}]$?
c) Let ϕ be an automorphism of L. What can you say about the restriction $\phi|_{\mathbb{Q}}$? What can you say about the restriction $\phi|_K$?
d) Find an element of order 4 in $\operatorname{Aut} L$. What is the group $\operatorname{Aut} L$ abstractly?
e) Replace $\sqrt{2}$ by $\sqrt{3}$, and $\sqrt{2+\sqrt{2}}$ by $\sqrt{3+\sqrt{3}}$. Try to redo parts (a) - (d). Do the same results still hold?

6. Find all algebraic field extensions of $\mathbb{R}$. Justify your assertions. (You may assume that $\mathbb{C} = \mathbb{R}[i]$ is algebraically closed.)

7. Let K be a field with algebraic closure $\bar{K}$. Let $K^{\mathrm{s}} = \{a \in \bar{K} \mid a \text{ is separable over } K\}$.
a) Show that K^{s} is a subfield of $\bar{K}$ (called the *separable closure* of K).
b) Show that for every separable polynomial $f(x) \in K[x]$, the field K^{s} contains a root of f, and $f(x)$ factors over K^{s} as the product of linear factors.
c) Show that K^{s} is normal over K.

8. a) Show that if K is an infinite field, and if $f(Y_1, \ldots, Y_n) \in K[Y_1, \ldots, Y_n]$ is a non-zero polynomial, then there exist $a_1, \ldots, a_n \in K$ such that $f(a_1, \ldots, a_n) \neq 0$. [Hint: Induction on n.]
b) What if K is finite?