Math 603 Problem Set #8 Due Monday, March 28, 2016

1. Let V be an affine variety with ring of functions R, over an algebraically closed field k. Let $W \subseteq V$ be a subvariety and let $I \subset R$ be a proper ideal. Prove or disprove each of the following equivalences. If only one implication in an equivalence is true, prove that one. If any implication is false, give a counterexample.

a) W is an irreducible closed subset of $V \Leftrightarrow I(W)$ is an irreducible ideal of R.

b) $V(I)$ is an irreducible closed subset of $V \Leftrightarrow I$ is an irreducible ideal of R.

2. Let k be a field.

a) Prove that a proper ideal in $k[x]$ is primary if and only if it is a power of a prime ideal.

b) Let $R = k[x,y,z]/(xy - z^2)$. Let $I = (x,z) \subset R$. Show that I is prime but I^2 is not primary. Do this explicitly by finding $a, b \in R$ such that $ab \in I^2$ but a is not in I^2 and neither is any power of b.

3. Let $\mathfrak{p}$ be a prime ideal in a commutative ring R.

a) Show that the symbolic power $\mathfrak{p}^{(n)}$ is primary, and that its associated prime is $\mathfrak{p}$.

b) Show that $\mathfrak{p}^n$ is primary if and only if $\mathfrak{p}^n = \mathfrak{p}^{(n)}$. Explain the relationship to problem 2(b).

4. Let n be a square-free non-zero integer. Let R_n be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$. Show that $R_n = \mathbb{Z}\left[\frac{1+\sqrt{n}}{2}\right]$ if $n \equiv 1 \pmod 4$, and that $R_n = \mathbb{Z}[\sqrt{n}]$ otherwise.

5. For each of the following rings R, determine whether R has a height one prime that is not principal. If there is one, find one explicitly. If there isn't one, determine whether there is *some* prime ideal that is not principal, and find one explicitly if it exists.

a) $\mathbb{Z}[i,x,y]$.

b) $\mathbb{Q}[x,y,z,w]/(xy - zw)$.

c) $\mathbb{Z}[\sqrt{-5}]$.

d) $\mathbb{Z}[x,y]/(5, y - x^3 - x + 1)$.

6. Let p be a prime number and let R be a commutative ring of characteristic p (i.e. $p \cdot 1 = 0$). Define the map $F : R \to R$ by $a \mapsto a^p$.

a) Show that F is a ring endomorphism (i.e. homomorphism from R to itself).

b) If R is a field, determine which elements lie in the set $\{a \in R \,|\, F(a) = a\}$. Do they form a ring? a field?

c) If R is a field, must F be injective? surjective? (Give a proof or counterexample for each.)

d) If R is a finite field, show that F is an automorphism.

7. Let K be a field and let G be a subgroup of the multiplicative group $K^\times = K - \{0\}$.

a) Show that if $a, b \in K$ have finite orders m, n, then there is a $c \in K$ whose order is the least common multiple of m, n. [Hint: First do the case of m, n relatively prime.]

b) Show that if G is finite then it is cyclic. [Hint: Let ℓ be the l.c.m. of the orders of the elements of G, and consider the roots of the polynomial $x^\ell - 1$.]

c) Conclude that if $K \subseteq L$ is an extension of finite fields, then $L = K[a]$ for some $a \in L$. [Hint: What is the group structure of $L^\times$?]