Math 603 Problem Set #7 Due Monday, March 21, 2016

1. a) Let R be a Noetherian ring, $\mathcal{I}$ the set of proper ideals of R, and $\mathcal{I}_0$ a subset of $\mathcal{I}$. Let P be a property that ideals in $\mathcal{I}_0$ may or may not have. Suppose that the following holds:

$\forall I \in \mathcal{I}_0$, if every ideal $J \in \mathcal{I}_0$ that properly contains I has property P, then so does I.

Conclude that P holds for *all* $I \in \mathcal{I}_0$.

b) Use this principle ("Noetherian induction") to prove that if R is a Noetherian integral domain, and $r \in R$ is a non-zero non-unit, then r is a product of irreducible elements of R. [Hint: What is $\mathcal{I}_0$?]

c) What does the principle say if $R = \mathbb{Z}$?

d) Show that (b) (and therefore (a)) fails in general if R is not Noetherian.

2. Determine the Krull dimensions of the following rings: $\mathbb{R}[x, x^{-1}]$, $\mathbb{C}[x, y, z]/(z^2 - xy)$, $\mathbb{Z}[x, y]/(y^2 - x)$, $\mathbb{Q}[x, y, z]/(y^2, z^3)$, $\mathbb{Q}[[x, y, z]]$, $\mathbb{Z}_{(2)}[x]$.

3. Given a commutative ring R, the *maximal spectrum* of R (denoted $\operatorname{Max} R$) is the set of maximal ideals of R. For each subset $E \subset R$, let $V(E) = \{\mathfrak{m} \in \operatorname{Max} R \mid E \subseteq \mathfrak{m}\} \subseteq \operatorname{Max} R$.

a) Show that $\operatorname{Max} R$ has a topology in which the closed sets are precisely the sets $V(E)$.

b) Show that $V(E) = V(I)$ for any $E \subset R$, where I is the ideal generated by E.

c) Show that $V(I) = V(\sqrt{I})$ for any ideal I.

d) Show that $V(\bigcup_\alpha E_\alpha) = \bigcap_\alpha V(E_\alpha)$ for any collection of subsets $\{E_\alpha\}_{\alpha \in A}$, and that $V(I_1 + \cdots + I_n) = V(I_1) \cap \cdots \cap V(I_n)$ for any ideals $I_1, \ldots, I_n$.

e) Show that $V(I_1 \cap \cdots \cap I_n) = V(I_1) \cup \cdots \cup V(I_n)$ for any ideals $I_1, \ldots, I_n$ of R. Also explain the relationship with problem 5 on Problem Set 6.

f) Give examples to illustrate (b) - (e) geometrically, for $R = \mathbb{R}[x]$.

g) If $R = \mathbb{C}[x, y]/(f)$, is there a continuous bijective map between $\operatorname{Max} R$ and the locus of zeroes of f in $\mathbb{C}^2$ (under the usual topology)? In which direction?

4. Consider the rings $R = \mathbb{C}[x]$, $\mathbb{C}[x, y]$, $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$, $\mathbb{C}[x, y]/(x^2 - y^2)$, $\mathbb{C}[x]/(x^2)$, $\mathbb{C}[x, y]/(x^2)$, $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $\mathbb{C}[x]/(x^2 - x)$, $\mathbb{Z}/2$, $\mathbb{Z}/6$, $\mathbb{Z}$, $\mathbb{Z}[1/15]$. For each of them, do the following:

a) Describe all the maximal ideals in R, and describe $\operatorname{Max} R$ geometrically.

b) Determine whether $\operatorname{Max} R$ is connected (in the topology given in problem 3).

5. a) Let $R = \mathbb{C}[x, y]/(x^2 - y^2)$ and $S = \mathbb{C}[x, y]/(x^2 - x)$. Show that there is a homomorphism $f : R \to S$ given by $f(x) = y - 2xy$, $f(y) = y$. Show that there is an induced continuous map $f^* : \operatorname{Max} S \to \operatorname{Max} R$ given by $\mathfrak{m} \mapsto f^{-1}(\mathfrak{m})$. Describe the map f^* geometrically. Is it injective? surjective? (A picture in the (x, y)-plane may help.)

b) In general, if $f : R \to S$ is a homomorphism of commutative rings, is there an induced continuous map $f^* : \operatorname{Max} S \to \operatorname{Max} R$? (What if $R = \mathbb{Z}$ and $S = \mathbb{Q}$?) What if we instead considered the *prime spectrum* of R and of S? (The prime spectrum $\operatorname{Spec} R$ is defined as the set of prime ideals of R with the topology defined similarly to that of Max.)

6. Let k be a field, and let $R = k[x, y]$.

a) Find the primary decompositions of the ideals $(y^2 - x^2)$ and $(y^2 - x^2)^2$. In each case, find the associated primes; determine whether the given ideal I is irreducible; and determine whether the closed subset determined by I is irreducible. [Caution: Be careful if $\operatorname{char} k = 2$.]

b) Consider the ideal $I = (x^2, xy)$ in R. Show that $(x) \cap (x^2, y)$ and $(x) \cap (x, y)^2$ are distinct primary decompositions of I. Find the associated primes for these two primary decompositions. Are there embedded primes? Is I irreducible? Is $V(I)$ an irreducible closed subset of 2-space over k?