1. a) Show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, then M is Noetherian if and only if M' and M'' are. [Hint: For the reverse implication, consider problem 2 of Problem Set 2 and use the Five Lemma.]

b) Deduce that R-modules N_1 and N_2 are each Noetherian if and only if $N_1 \oplus N_2$ is.

c) Let S be a multiplicative subset of a commutative ring R. Show that if R is Noetherian then so is the localization $S^{-1}R$. Does the converse also hold?

2. a) Do there exist infinite strictly increasing chains $I_1 \subset I_2 \subset \cdots$ of ideals in $\mathbb{C}[x]$? Do there exist finite strictly increasing chains $I_1 \subset I_2 \subset \cdots \subset I_n$ of ideals in $\mathbb{C}[x]$ with arbitrarily large n?

b) Repeat part (a), but with strictly *decreasing* chains of ideals $I_1 \supset I_2 \supset \cdots$ rather than strictly increasing chains.

c) Explain geometrically your assertions in parts (a) and (b).

3. Modify the proof of the Hilbert Basis Theorem to prove that if R is Noetherian then so is $R[[x]]$.

4. Show that the analogs of parts (a) and (b) of problem 1 above hold in the case of Artin modules (modules whose submodules satisfy the descending chain condition).

5. a) Let $V \subseteq \mathbb{C}^n$ be a complex affine variety, given by the common zeroes of some polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$, with ring of functions $R = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let W be a non-empty Zariski closed subset of V, i.e. the set of common zeroes of some elements of R. Let $I = I(W) \subset R$, the ideal of functions that vanish on all of W. Show that W is irreducible if and only if I is a prime ideal.

b) Let $I_1, \ldots, I_n$ be proper ideals of R, and for each i let $W_i = V(I_i)$, the variety defined by I_i (i.e. the set of common zeroes of the elements of I_i). Show that

$$V(I_1 + \cdots + I_n) = W_1 \cap \cdots \cap W_n,$$

$$V(I_1 \cap \cdots \cap I_n) = W_1 \cup \cdots \cup W_n.$$

Also explain the relationship with problem 6 on Math 602 (fall 2015) Problem Set 9.

6. a) Let R be a Noetherian ring and $I \subset R$ an ideal. Prove that there are only finitely many prime ideals that are minimal over I (i.e. are minimal elements of the set of prime ideals that contain I). [Hint: If not, show that there is a maximal counterexample I, and that trivially this I is not prime. Show that if $a, b \in R - I$ with $ab \in I$, then every prime that is minimal over I is also minimal over either $I + (a)$ or $I + (b)$.]

b) Deduce that every Noetherian ring has finitely many minimal primes. Also, interpret this assertion geometrically, if R is the ring of functions on an affine variety V (i.e. a subset of n-space defined by the vanishing of some polynomials).

c) What happens in (a) and (b) if the ring is not Noetherian?