Math 603 Problem Set #5 Due Monday, Feb. 29, 2016

1. Let M be an R-module. Prove that the following conditions on M are equivalent:

i) M is faithfully flat (i.e. for every sequence of R-modules $N' \to N \to N''$, the given sequence is exact iff $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact).

ii) For every *complex* of R-modules $N' \to N \to N''$, the given complex is exact iff $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact.

iii) M is flat; and for every homomorphism of R-modules $\phi : N_1 \to N_2$, ϕ is surjective iff $1 \otimes \phi : M \otimes N_1 \to M \otimes N_2$ is surjective.

2. Let A be a flat R-algebra. Prove that the following conditions on A are equivalent:

i) A is faithfully flat over R.

ii) Every maximal ideal of R is the contraction of a maximal ideal of A.

iii) $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective.

iv) For every R-module N, if $A \otimes_R N = 0$ then $N = 0$.

[See the webpage for hints, if needed.]

3. For each of the following R-algebras A, determine whether A is a finitely generated R-module and whether it a finitely generated R-algebra. Also determine whether the R-module A is flat and whether it is faithfully flat.

a) Let $R = \mathbb{Z}$. Take $A = \mathbb{Z}[x]/(3x)$, $\mathbb{Z}[1/5]$, $\mathbb{Z}[i]$, $\mathbb{Z}[i, 1/5]$, $\mathbb{Z}[i, 1/(2+i)]$.

b) Let $A = \mathbb{R}[[x]]$. Take $R = \mathbb{R}$, $\mathbb{R}[x]$, $R = \mathbb{R}[x]_{(x)}$.

4. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be given by the matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

and let M be the cokernel of ϕ.

a) Find all $n \in \mathbb{Z}$ such that the $\mathbb{Z}$-module M has (non-zero) n-torsion.

b) Is M free? flat? torsion free? projective?

c) Show that M has a finite free resolution.

d) For each prime number p, compute $M \otimes \mathbb{Z}/p = \operatorname{Tor}^0(M, \mathbb{Z}/p)$ and $\operatorname{Tor}^1(M, \mathbb{Z}/p)$.

e) For every $\mathbb{Z}$-module N and every $i \geq 2$, compute $\operatorname{Tor}^i(M, N)$.

5. Let M, N be R-modules, and let $0 \to N \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_2 \xrightarrow{f_2} \cdots$ be an injective resolution of N. Let $\phi \in \operatorname{Ext}^1(M, N)$, and choose a homomorphism $\Phi \in \operatorname{Hom}(M, I_1)$ representing ϕ (where we use the above resolution to compute Ext).

a) Show that $f_1 \circ \Phi = 0$, and deduce that Φ factors through a map $M \to I_0/N$.

b) Let $M' = M \times_{I_0/N} I_0$, the fiber product of R-modules taken with respect to the above map $M \to I_0/N$ and the reduction map $I_0 \to I_0/N$. Show that the first projection map $M' \to M$ is surjective and that its kernel is N.

c) We define $\operatorname{Ext}(M, N)$ to be the set of equivalence classes of extensions $0 \to N \to L \to M \to 0$ of M by N. Explain how the above map $M' \to M$ induces an element of $\operatorname{Ext}(M, N)$. This element is denoted by $c(\phi)$.

d) Show that $c : \operatorname{Ext}^1(M, N) \to \operatorname{Ext}(M, N)$ defines a bijection.