Math 603 Problem Set #4 Due Mon., Feb. 22, 2016

1. Let R be a commutative ring, and let M, N, S be R-modules. Assume that M is finitely presented and that S is flat. Consider the natural map

$$\alpha : S \otimes_R \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, S \otimes_R N)$$

taking $s \otimes \phi$ (for $s \in S$ and $\phi \in \operatorname{Hom}(M, N)$) to the homomorphism $m \mapsto s \otimes \phi(m)$.

a) Show that if M is a free R-module then α is an isomorphism. [Hint: If $M = R^n$, show that both sides are just $(S \otimes_R N)^n$.]

b) Suppose more generally that $R^a \to R^b \to M \to 0$ is a finite presentation for M. Show that the induced diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & S \otimes_R \operatorname{Hom}(M, N) & \longrightarrow & S \otimes_R \operatorname{Hom}(R^b, N) & \longrightarrow & S \otimes_R \operatorname{Hom}(R^a, N) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \operatorname{Hom}(M, S \otimes_R N) & \longrightarrow & \operatorname{Hom}(R^b, S \otimes_R N) & \longrightarrow & \operatorname{Hom}(R^a, S \otimes_R N)
\end{array}$$

is commutative and has exact rows.

c) Using the Five Lemma and part (a), deduce that α is an isomorphism.

2. Let M and N be finitely generated modules over a commutative ring R, such that $M \otimes_R N = 0$.

a) Show that if R is a local ring with maximal ideal $\mathfrak{m}$, then M or N is 0. [Hint: Nakayama's Lemma.]

b) What if R is not local?

3. Let R be a commutative ring with Jacobson radical J. For $x \in R$, let P_x be the set of $r \in R$ such that $r \equiv 1 \pmod{x}$ (i.e. such that $r - 1 \in xR$). Let $R^\times$ denote the multiplicative group of units in R.

a) Show that $J = \{x \in R \mid P_x \subseteq R^\times\}$.

b) Let M be a finitely generated R-module and let $a_1, \dots, a_n \in M$, where $n \geq 0$. Show that the R-module M is generated by $a_1, \dots, a_n$ if and only if the R/J-module M/JM is generated by the images of these elements. [Hint: Either generalize the proof of the local case, or else reduce to that case.]

c) Deduce that if M is a finitely generated R-module and $M/JM = 0$ then $M = 0$. Explain why this generalizes Nakayama's Lemma.

4. Let M be a finitely generated R-module. Prove that the following conditions on M are equivalent:

i) M is locally free over R (i.e. $M_\mathfrak{m}$ is free over $R_\mathfrak{m}$ for all maximal ideals $\mathfrak{m} \subset R$).

ii) For every maximal ideal $\mathfrak{m} \subset R$, there is an $f \notin \mathfrak{m}$ such that M_f is free over R_f.

iii) $\exists\, f_1, \dots, f_n \in R$ such that $(f_1, \dots, f_n) = 1$ and M_{f_i} is free over R_{f_i} for $i = 1, \dots, n$.

[Hint: In (ii) $\Rightarrow$ (iii), what ideals contain the set $S = \{f \in R \mid M_f \text{ is free over } R_f\}$?]