1. Let P be a finitely generated projective R-module.

a) Show that there is a finitely generated free R-module F, and an R-module K, such that $0 \to K \stackrel{i}{\to} F \stackrel{\pi}{\to} P \to 0$ is exact.

b) Show that there exists a homomorphism $j : F \to K$ such that i is a section of j, and that the sequence $F \stackrel{i \circ j}{\to} F \stackrel{\pi}{\to} P \to 0$ is exact.

c) Conclude that P is a finitely presented R-module.

d) Give an example of a finitely generated R-module M (for some R) that is *not* projective and is not finitely presented.

2. Let R be a commutative ring.

a) Show that $R[x]$ is a flat R-module.

b) Show that $R[x]/(rx)$ is *not* a flat R-module if R is an integral domain and r is a non-zero non-unit element of R.

c) Let M, N be flat R-modules. Show that $M \oplus N$ and $M \otimes_R N$ are flat R-modules. Do the converses hold also?

d) Show that if M is a projective R-module, then M is a flat R-module. (Hint: Use part of (c).)

e) Is the $\mathbb{Z}$-module $\mathbb{Q}$ free? torsion free? flat? projective?

3. In the notation of Problem Set 1, problems 4 and 5:

a) Find linear polynomials $f, g \in R$ such that the only maximal ideal of R containing f is I, and the only maximal ideal of R containing g is J. (Hint: Where can the graphs of $f = 0$ and of $g = 0$ intersect the circle?)

b) Find a linear polynomial $h \in R$ such that $h \in J$ and $h \in K$, where K is the maximal ideal corresponding to the point $S = (3, -4)$.

c) Show that $I_f \stackrel{\text{def}}{=} I \otimes_R R[\frac{1}{f}]$ is a free $R[\frac{1}{f}]$-module, viz. is the unit ideal in $R[\frac{1}{f}]$. (Hint: Show $f \in I_f$.)

d) Show that for suitable choice of g, h above, $\dfrac{x-3}{y-4} = \dfrac{h}{g}$ in R. Explain this equality geometrically, in terms of the graphs of $g = 0$, $h = 0$, $x - 3 = 0$, $y - 4 = 0$, and $x^2 + y^2 = 25$.

e) Using (d), show that $I_g \stackrel{\text{def}}{=} I \otimes_R R[\frac{1}{g}]$ is a free $R[\frac{1}{g}]$-module, viz. is the ideal $(y - 4)$ in $R[\frac{1}{g}]$. Explain why this is reasonable geometrically.

4. a) Let R be a commutative ring and suppose that *every* R-module M is free. Show that R is a field.

b) Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 25)$. Is R a PID? Is every finitely generated projective R-module free?

5. Let M be an R-module and let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. Under each of the following four conditions (considered separately), either show that the sequence $0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$ must be exact or else give a counterexample.

a) M is flat.

b) N' is flat.

c) N is flat.

d) N'' is flat.