Math 603 Problem Set #2 Due Mon., Feb. 8, 2016

1. Suppose that

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

is a commutative diagram of R-modules, with exact rows.

a) Show that if α_1 is surjective and α_2, α_4 are injective, then α_3 is injective.

b) Show that if α_5 is injective and α_2, α_4 are surjective, then α_3 is surjective.

c) In particular, deduce that α_3 is an isomorphism provided that $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are.

(The above result is the strong version of the "Five Lemma", which is named after the appearance of this diagram.)

2. Suppose that $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules. Let $M_1 \subseteq M_2 \subseteq \cdots$ be a chain of R-submodules of M, and define $M_i' = f^{-1}(M_i)$ and $M_i'' = g(M_i)$.

a) Show that $M_1' \subseteq M_2' \subset \cdots$ is a chain of R-submodules of M'.

b) Show that $M_1'' \subseteq M_2'' \subset \cdots$ is a chain of R-submodules of M''.

c) Show that if $i < j$, then the inclusion map $M_i \hookrightarrow M_j$ induces inclusions $M_i' \hookrightarrow M_j'$ and $M_i'' \hookrightarrow M_j''$, and also induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i' & \longrightarrow & M_i & \longrightarrow & M_i'' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M_j' & \longrightarrow & M_j & \longrightarrow & M_j'' & \longrightarrow & 0 \end{array}$$

3. In the notation of problems 4 and 5 of Problem Set 1:

a) Show that $I \cap J = (y-4)$ and $I + J = (1)$ in R.

b) Let $\Delta : I \cap J \to I \oplus J$ be given by $\Delta(f) = (f, f)$, and let $- : I \oplus J \to I + J$ be given by $-(f, g) = f - g$. Show that the following sequence of R-modules is exact:

$$0 \to I \cap J \xrightarrow{\Delta} I \oplus J \xrightarrow{-} I + J \to 0 \qquad (*)$$

c) Show that the exact sequence $(*)$ is split. (Hint: What is $I + J$?)

d) Deduce that $I \oplus J$ is isomorphic to $(I \cap J) \oplus (I + J)$, and conclude that $I \oplus J$ is therefore free of rank 2 (thereby giving another proof of problem 5(c) of Problem Set 1).

4. In the notation of the above problem:

a) Explicitly find a section s of $-$, corresponding to a splitting of the exact sequence $(*)$. (Hint: Find $(a, b) \in I \oplus J$ such that $a - b = 1 \in R$.)

b) Explicitly find an isomorphism $\alpha : (I \cap J) \oplus (I + J) \to I \oplus J$ induced by the section in (a).

c) Show that problem 5 of Problem Set 1 also yields an isomorphism

$$(I \cap J) \oplus (I + J) \to I \oplus J.$$

(Hint: Use problem 3(a) above.) Then compare that isomorphism with the isomorphism α in part (b) above.