1. a) Which of the following $\mathbb{Z}$-modules are finitely generated? Which are free? Which are $\mathbb{Z}$-algebras? Among the $\mathbb{Z}$-algebras, which are finitely generated as $\mathbb{Z}$-algebras? $\mathbb{Z}/5 \times \mathbb{Z}/7$, $5\mathbb{Z}$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\pi]$, $\mathbb{Z}[1/2]$, $\mathbb{Q}$, $\frac{1}{2}\mathbb{Z} = \{\frac{n}{2} \mid n \in \mathbb{Z}\}$, $\mathbb{Z}[x]/(2x)$.

b) Do the same with $\mathbb{Z}$ replaced by $\mathbb{R}[x]$, for these modules: $\mathbb{R}[x,y]$, $\mathbb{R}[x,y]/(y^2-x)$, $\mathbb{R}[x,y]/(xy-1)$, $\mathbb{R}[x,y]/(xy)$.

2. a) If R is a commutative ring, find a ring isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \xrightarrow{\sim} R[x]/(x^2-2)$.

b) Determine whether $\mathbb{Z}[\sqrt{3}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ are integral domains.

c) Simplify each of the following $\mathbb{Z}$-modules (up to isomorphism):
$\mathrm{Hom}(\mathbb{Z}/10,\mathbb{Z})$ = the dual of the $\mathbb{Z}$-module $\mathbb{Z}/10$, $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}/10)$, $\mathrm{Hom}(\mathbb{Z}/10,\mathbb{Z}/6)$, $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/10$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Z}/6$, $\mathbb{Z}/10 \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \mathbb{Z}[1/5]$, $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

3. Let k be a field. Let $R = k[z]$, $f(x) \in A = k[x]$, and $g(y) \in B = k[y]$, with $f, g \notin k$.

a) Explain why there are unique injections of k-algebras $i : R \hookrightarrow A$ and $j : R \hookrightarrow B$ such that $i(z) = f(x)$, $j(z) = g(y)$. Then explain how these maps make A, B into R-algebras (and not just k-algebras), and hence also make $A \otimes_R B$ into an R-algebra.

b) Show that the set of k-algebra homomorphisms $A \otimes_R B \to k$ is in natural bijection with the elements of the set $S = \{(a,b) \in k^2 \mid f(a) = g(b)\}$.

c) Give a geometric interpretation to your answer to part (b), in terms of fiber product.

4. Let R be the ring of real polynomial functions on the circle $x^2 + y^2 = 25$. Let P be the point $(3,4)$, and let I be the ideal of functions in R vanishing at P.

a) Show that I is generated by the elements $x-3$, $y-4$.

b) Show that if $I = (f)$ where $f \in R$, then f divides $x-3$ and $y-4$ in R. Deduce that f cannot vanish at any point of the circle except for P. Also, deduce that f cannot vanish to order ≥ 2 at P, as a function on the circle.

c) Deduce that I cannot be principal. (Hint: Let $F(x,y) \in \mathbb{R}[x,y]$ represent the function $f \in R$. What does the graph of $F(x,y) = 0$ look like? Where does it meet the circle? Is it tangent or transversal to the circle?)

d) Show that no two elements in I are linearly independent over the ring R.

e) Using (c) and (d), conclude that I is not a free R-module.

5. In the notation of problem 4, let Q be the point $(-3,4)$, and let J be the ideal of functions in R vanishing at Q. Consider the R-submodule $I \oplus J$ of $R \oplus R$.

a) Show that every element of $R \oplus R$ of the form

$$\big((x-3)f + (y-4)g,\ (x+3)f + (y-4)g\big),$$

for some $f, g \in R$, lies in $I \oplus J$.

b) Show conversely every element $(i,j) \in I \oplus J$ can be uniquely expressed in the above form for some $f, g \in R$. (Hint: First solve for f, g as linear combinations of i, j. Then use 4(a) and the identity $x^2 + y^2 = 25$ in R, to show that $\frac{x+3}{y-4} i \in R$; and similarly treat $\frac{x-3}{y-4} j$.)

c) Deduce that $I \oplus J$ is a free R-module of rank 2, even though I is not free. (Is J free?)