1. Call $T \in \operatorname{End}(V)$ an *idempotent* if $T^2 = T$. Show that if V is finite dimensional and T is an idempotent, then there are subspaces $X, Y \subset V$ such that $V = X \times Y$, $T|_X = 0$, $T|_Y =$ identity. Deduce that with respect to some basis of V, the idempotent map T is given by a diagonal matrix whose diagonal entries are of the form $(1, 1, \dots, 1, 0, 0, \dots, 0)$.

2. a) For K a field, suppose that $A \in M_n(K)$ is strictly upper triangular (i.e. A is upper triangular, and the diagonal entries are all 0). Show that A is nilpotent, and find the index of nilpotence (i.e. the minimal m such that $A^m = 0$).

b) Show that if S and T are upper triangular, then their *bracket* $[S, T] := ST - TS$ is nilpotent.

c) Let A_0 be the set of upper triangular matrices in $M_n(K)$. Show that A_0 is a *Lie algebra*, in the sense that it is closed under addition, scalar multiplication, and bracket. Also, inductively define A_i by $A_{i+1} = [A_i, A_i] = \langle [S, T] \,|\, S, T \in A_i \rangle$. Show that some $A_r = 0$. (A_0 is thus called *solvable*, in analogy with the fact that a finite group is solvable iff its successive commutators terminate in the trivial group.)

3. Let V be a vector space and let $G \subseteq \operatorname{Aut} V$ be a finite subgroup. Say that $W \subseteq V$ is *G-invariant* if it is T-invariant for every $T \in G$. Say that $W \subseteq V$ is *G-irreducible* if W is G-invariant and the only G-invariant subspaces of W are 0 and W.

a) If V is a finite dimensional vector space over a field of characteristic zero, show that V can be written as the direct product of G-irreducible subspaces. [Hint: Generalize the argument in problem 5 of PS11.]

b) What can you say about the conclusion of part (a) if the field of scalars is not necessarily of characteristic zero?

4. Let R be the ring of polynomial functions on the unit sphere $S^2 \subset \mathbb{R}^3$. Thus this ring is given by $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

a) Let $P = (0, 0, 1) \in S^2$, and let $R_P = \{\frac{f}{g} \,|\, f, g \in R;\ g(P) \neq 0\}$. Show directly that R_P is a local ring (i.e. has exactly one maximal ideal I), and find a set of generators for I.

b) Show that $I^2 \subset I$ but that $I^2 \neq I$. Let I/I^2 be the image of I under the ring homomorphism $R_P \to R_P/I^2$. Show that I/I^2 is a 2-dimensional vector space over $\mathbb{R}$. [Hint: Find a basis, using that $z - 1 = \frac{-1}{z+1} \cdot (x^2 + y^2) \in I^2$.]

5. In the situation of problem 4:

a) Let $T \subset \mathbb{R}^3$ be the tangent plane to S^2 at P. Thus $T = \{(x, y, 1) \,|\, x, y \in \mathbb{R}\}$. Show that T is a 2-dimensional vector space over $\mathbb{R}$, under the addition $(x, y, 1) + (x', y', 1) = (x + x', y + y', 1)$ and scalar multiplication $c(x, y, 1) = (cx, cy, 1)$. What is the 0-vector?

b) Let $f \in T^*$, the dual space of T. Show that $f : T \to \mathbb{R}$ extends to a unique linear functional $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$. Let $\bar{f} : S^2 \to \mathbb{R}$ be the restriction of $\tilde{f}$ to S^2. Show that $\bar{f} \in R$, and moreover $\bar{f} \in I$.

c) If $f \in T^*$, let $\phi(f) \in I/I^2$ be the image of $\bar{f} \in I$ under $I \to I/I^2$. Show that $\phi : T^* \to I/I^2$ is an isomorphism of vector spaces.

d) Conclude that T is isomorphic to $(I/I^2)^*$ via ϕ^*.

Remark. This problem works more generally for any smooth space $S \subseteq \mathbb{R}^n$ defined by polynomials. Often, geometers turn this problem on its head and *define* the tangent space to be $(I/I^2)^*$. The advantage is that this makes T intrinsic to S, rather than depending on the way that S is embedded in $\mathbb{R}^n$.