Math 602 Problem Set #4 Due Friday., Oct. 8, 2021

1. a) Assume that $\#G = pq$, where p and q are prime. Show that at least one of its Sylow subgroups is normal.

b) With G as above, assume $p \geq q$. Show that either G is abelian or else q divides $p-1$.

c) Find all groups of order 51, and all groups of order 55. Which are simple? solvable? nilpotent? abelian? cyclic?

2. Find an extension G of C_6 by C_7 such that the generator of C_6 acts by conjugation on C_7 as an automorphism of order 3. How many Sylow p-subgroups does G have, for each p?

3. Let G be a p-group, and let Φ be its Frattini subgroup.

a) Show that if $g \in G$ then $g^p \in \Phi$. (Hint: If $H \subset G$ is a maximal subgroup, show that $g^p \in H$ by considering its image in G/H.)

b) Deduce that every element of G/Φ has order 1 or p.

c) Conclude that G/Φ is isomorphic to $(\mathbb{Z}/p)^n = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ (with n factors) for some $n \geq 0$.

4. If K is a group and S is a subset of K that generates K, we will call S a *minimal generating set* for K if no proper subset of S also generates K.

a) Show that every minimal generating set of $(\mathbb{Z}/p)^n$ has exactly n elements. (Hint: View $(\mathbb{Z}/p)^n$ as a vector space.)

b) Prove or disprove: If G is any finite group, then any two minimal generating sets for G have the same number of elements.

c) Let G be a p-group with Frattini subgroup Φ, so that G/Φ is isomorphic to $(\mathbb{Z}/p)^n$ (as in problem 1(c)). Show that

(i) Every minimal generating set for G has exactly n elements.

(ii) If T is a subset of G with exactly n elements, then T is a minimal generating set for G if and only if its image under $G \twoheadrightarrow G/\Phi$ is a minimal generating set for G/Φ.

(Hint: Use Problem Set 2 #4 and problems 3(c) and 4(a) above.)

(Remark: Part (c) is also called the Burnside Basis Theorem.)

5. a) Show that every element of A_5 is conjugate (in A_5) to exactly one of the following five elements:

$$1, (123), (12)(34), (12345), (12354).$$

Determine the number of elements conjugate to each.

b) Deduce that A_5 is simple. [Hint: Show that every normal subgroup is a union of conjugacy classes. Then apply part (a) and Lagrange's Theorem.]

c) Show that neither A_5 nor S_5 is solvable.