1. Consider an action of a group G on a set X.

a) Show by example that the stabilizers of two elements of X can have different orders.

b) Show that if $x_1, x_2 \in X$ lie in the same orbit, then their stabilizers must have the same order, and in fact must be conjugate subgroups.

c) Must the stabilizers of x_1, x_2 in (b) be equal? Must the stabilizer of an element of X be normal in G?

2. Interpret each of the following objects in terms of stabilizers, and determine which must be normal subgroups of G:

a) The kernel of a group action $\phi : G \to \mathrm{Sym}(X)$.

b) The centralizer of an element $x \in G$.

c) The center of a group G.

d) The normalizer $N_G(H)$ of a subgroup $H \subseteq G$.

e) The centralizer $C_G(H)$ of a subgroup $H \subseteq G$.

3. Given a group G acting on a finite set X, and an element $x \in X$, write Gx for the orbit of x and write G_x for the stabilizer of x.

a) Show that $|Gx| = (G : G_x)$. (In particular, the right hand side is finite.)

b) Consider the orbits that have more than one element, pick one element from each of these orbits, and gather them together as a set S. Show that $|X| = |X^G| + \sum_{x \in S}(G : G_x)$, where $X^G \subseteq X$ is the set of elements that are fixed by all of G.

c) Interpret these two equalities in each of these two cases, where G is a finite group:

i) G acts on itself by conjugation.

ii) A subgroup K of G acts on the set of subgroups of G by conjugation.

4. a) If H is a subgroup of G and $H \neq G$, we say that H is a *maximal* subgroup if the only subgroups containing H are itself and G. Show that if H is maximal then so is aHa^{-1}, for any $a \in G$.

b) Define the *Frattini* subgroup Φ of G to be the intersection of the maximal subgroups of G. Show that $\Phi \triangleleft G$.

c) Find the Frattini subgroup Φ of the groups D_4, C_4, and $C_4 \times C_2 \times C_2$. Do the same for the *quaternion group* $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (under the usual multiplication of quaternions). In each case, find G/Φ. Conjecture?

5. Prove or disprove each of the following.

a) A group is abelian if and only if every subgroup is normal.

b) Let H, K be subgroups of G. Then $HK := \{hk \,|\, h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$. This equality holds in particular if either H or K is normal.