1. In the situation of PS11, problem 5:

a) Let $T \subset \mathbb{R}^3$ be the tangent plane to S^2 at P. Thus $T = \{(x, y, 1) \,|\, x, y \in \mathbb{R}\}$. Show that T is a 2-dimensional vector space over $\mathbb{R}$, under the addition $(x, y, 1) + (x', y', 1) = (x + x', y + y', 1)$ and scalar multiplication $c(x, y, 1) = (cx, cy, 1)$. What is the 0-vector?

b) Let $f \in T^*$, the dual space of T. Show that $f : T \to \mathbb{R}$ extends to a unique linear functional $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$. Let $\bar{f} : S^2 \to \mathbb{R}$ be the restriction of $\tilde{f}$ to S^2. Show that $\bar{f} \in R$, and moreover $\bar{f} \in I$.

c) If $f \in T^*$, let $\phi(f) \in I/I^2$ be the image of $\bar{f} \in I$ under $I \to I/I^2$. Show that $\phi : T^* \to I/I^2$ is an isomorphism of vector spaces.

d) Conclude that T is isomorphic to $(I/I^2)^*$ via ϕ^*.

Remark. This problem works more generally for any smooth space $S \subseteq \mathbb{R}^n$ defined by polynomials. Often, geometers turn this problem on its head and *define* the tangent space to be $(I/I^2)^*$. The advantage is that this makes T intrinsic to S, rather than depending on the way that S is embedded in $\mathbb{R}^n$.

2. Let V be a finite dimensional vector space over a field K of characteristic zero, and let $T \in \operatorname{End} V$. If $W \subseteq V$, call W a *T-irreducible* subspace if W is T-invariant (i.e. $T(W) \subseteq W$) and the only T-invariant subspaces of W are 0 and W.

a) Suppose that $T \in \operatorname{End}(V)$ has order n in $\operatorname{End}(V)$ under composition, and that $W \subseteq V$ is a T-invariant subspace. Show that W has a T-invariant complement W'. [Hint: Pick an arbitrary complement W'', i.e. $V = W \times W''$. Let $P : V = W \times W'' \to W$ be the first projection map, and define $S : V \to V$ by $v \mapsto \frac{1}{n} \sum_{i=0}^{n-1} T^i P T^{-i}(v)$. Show $S^2 = S$. Then consider $\ker S$ and $\operatorname{im} S$.]

b) Under the hypotheses of (a), show that V can be written as the direct product of T-irreducible subspaces.

c) What if T does *not* have finite order in $\operatorname{End}(V)$?

d) What if K does *not* have characteristic zero?

3. Let V be a vector space and let $G \subseteq \operatorname{Aut} V$ be a finite subgroup. Say that $W \subseteq V$ is *G-invariant* if it is T-invariant for every $T \in G$. Say that $W \subseteq V$ is *G-irreducible* if W is G-invariant and the only G-invariant subspaces of W are 0 and W.

a) If V is a finite dimensional vector space over a field of characteristic zero, show that V can be written as the direct product of G-irreducible subspaces. [Hint: Generalize the argument in problem 2.]

b) What can you say about the conclusion of part (a) if the field of scalars is not necessarily of characteristic zero?

4. a) Consider the quadratic form $q = \langle 1, 1 \rangle$ over $\mathbb{Q}$. For which $c \in \mathbb{Q}^\times$ is the form $c \cdot q := \langle c, c \rangle$ isometric to q? [Hint: Which values does $c \cdot q$ take on?]

b) Let $h = \langle 1, -1 \rangle$ over a field K of characteristic $\neq 2$. Show that for every $c \in K^\times$, the form $c \cdot h := \langle c, -c \rangle$ is isometric to h, and so is hyperbolic. [Hint: After a change of variables, h becomes xy. What about $c \cdot h$?]

c) Let q be the quadratic form $\langle 1, 1, 1 \rangle$ over the field of 3 elements. Find the Witt decomposition of q explicitly.