1. Let n be an integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $P_n \subset \mathbb{R}[x]$ be the vector space of polynomials of degree $\leq n$. Define $F : P_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.

a) Show that F is an isomorphism. [Hint: $\dim P_n =?$ $\ker F =?$]

b) *Explicitly* find $F^{-1}(e_1), \ldots F^{-1}(e_{n+1})$ (where $e_1, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{R}^{n+1}$) in the case $n = 3$, $\alpha_j = j$ ($j = 1, 2, 3, 4$). [Hint: where does the polynomial $(x-a)(x-b)(x-c)$ vanish?]

c) Deduce that $F^{-1}(e_1), \ldots F^{-1}(e_{n+1})$ form a basis of P_n. In the case considered in (b), express x as a linear combination of them.

2. Call $T \in \mathrm{End}(V)$ an *idempotent* if $T^2 = T$. Show that if V is finite dimensional and T is an idempotent, then there are subspaces $X, Y \subset V$ such that $V = X \times Y$, $T|_X = 0$, $T|_Y =$ identity. Deduce that with respect to some basis of V, the idempotent map T is given by a diagonal matrix whose diagonal entries are of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$.

3. Let V, W, Y be finite dimensional vector spaces over K.

a) Show that there are natural isomorphisms $(V \otimes W)^* = V^* \otimes W^* = \mathrm{Hom}(V, W^*) = \mathrm{Hom}(W, V^*)$.

b) Show that there is a natural isomorphism $\mathrm{Hom}(V \otimes W, Y) = \mathrm{Hom}(V, \mathrm{Hom}(W, Y))$.

c) Show that $\mathrm{Hom}(V \otimes W, Y)$ is naturally isomorphic to the vector space of bilinear maps $V \times W \to Y$.

4. a) Let V be a K-vector space and let $0 \to W' \to W \to W'' \to 0$ be an exact sequence of K-vector spaces. Show that the induced sequences $0 \to V \otimes W' \to V \otimes W \to V \otimes W'' \to 0$; $0 \to \mathrm{Hom}(V, W') \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W'') \to 0$; and $0 \to \mathrm{Hom}(W'', V) \to \mathrm{Hom}(W, V) \to \mathrm{Hom}(W', V) \to 0$ are also exact. [Hint: Choose a basis for V.]

b) What if instead we consider modules over a ring R?

5. Let R be the ring of polynomial functions on the unit sphere $S^2 \subset \mathbb{R}^3$. Thus this ring is given by $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

a) Let $P = (0, 0, 1) \in S^2$, and let $R_P = \{\frac{f}{g} \mid f, g \in R;\ g(P) \neq 0\}$. Show directly that R_P is a local ring (i.e. has exactly one maximal ideal I), and find a set of generators for I.

b) Show that $I^2 \subset I$ but that $I^2 \neq I$. Let I/I^2 be the image of I under the ring homomorphism $R_P \to R_P/I^2$. Show that I/I^2 is a 2-dimensional vector space over $\mathbb{R}$. [Hint: Find a basis, using that $z - 1 = \frac{-1}{z+1} \cdot (x^2 + y^2) \in I^2$.]

6. a) For K a field, suppose that $A \in M_n(K)$ is strictly upper triangular (i.e. A is upper triangular, and the diagonal entries are all 0). Show that A is nilpotent, and find the index of nilpotence (i.e. the minimal m such that $A^m = 0$).

b) Show that if S and T are upper triangular, then their *bracket* $[S, T] := ST - TS$ is nilpotent.

c) Let A_0 be the set of upper triangular matrices in $M_n(K)$. Show that A_0 is a *Lie algebra*, in the sense that it is closed under addition, scalar multiplication, and bracket. Also, inductively define A_i by $A_{i+1} = [A_i, A_i] = \langle [S, T] \mid S, T \in A_i \rangle$. Show that some $A_r = 0$. (A_0 is thus called *solvable*, in analogy with the fact that a finite group is solvable iff its successive commutators terminate in the trivial group.)