1. If $I, J \subseteq R$ are ideals in a commutative ring, define the *ideal quotient* $(I : J) \subseteq R$ to be $\{a \in R \,|\, aJ \subseteq I\}$. Show that this is an ideal. If $R = \mathbb{Z}$, prove that $((m) : (n)) = (m/\gcd(m, n))$.

2. If $R \subseteq S$ are commutative rings and $I \subseteq R$ is an ideal of R, call $IS \subseteq S$ the *extension* of I to S. If $J \subseteq S$ is an ideal of S, call $J \cap R \subseteq R$ the *contraction* of J to R.

a) Are extensions and contractions of ideals always ideals? What about proper ideals? Are extension and contraction inverse operations?

b) For which prime ideals of $\mathbb{Z}$ is the extension to $\mathbb{Z}[i]$ also prime? For those that are not, which extensions are the product of two distinct prime ideals, and which are the square of a prime ideal of $\mathbb{Z}[i]$? (Of these two cases, the former case is called *split* and the latter case is called *ramified.*)

c) Show that taking contraction induces a surjection from the prime ideals of $\mathbb{Z}[i]$ to the prime ideals of $\mathbb{Z}$. Is it injective?

d) Do your assertions in part (c) hold for an arbitrary extension of integral domains $R \subseteq S$?

3. a) Let $V = \{\text{differentiable functions on } \mathbb{R}\}$. Prove that the functions e^x, e^{2x}, e^{3x} are linearly independent in the real vector space V. [Hint: If not, differentiate twice.]

b) Let W be the set of solutions to the differential equation $f'' - f = 0$, and let V be the set of solutions to $f''' - f' = 0$. Show that W is a vector subspace of V, find a basis for W, and extend this basis to a basis of V.

4. a) If V and W are vector spaces over a field K, and if $F : V \to W$ is a homomorphism, let $F^* : W^* \to V^*$ be the map on dual spaces given by $F^*(\phi) = \phi \circ F$. Show that $F \mapsto F^*$ defines a homomorphism $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$. Show that this homomorphism is natural, in the sense that $(F \circ G)^* = G^* \circ F^*$ if $F : V \to W$, $G : U \to V$.

b) Show that the above map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$ is an isomorphism if V and W are finite dimensional.

c) Show that if $0 \to U \to V \to W \to 0$ is exact, then so is $0 \to W^* \to V^* \to U^* \to 0$.

d) What if instead we consider modules over a ring R?

5. For any finite dimensional vector space V with basis $B = \{e_1, \ldots, e_n\}$, and dual basis $B^* = \{\delta_1, \ldots, \delta_n\}$ of V^*, define $\phi_{V,B} : V \to V^*$ by $\sum_1^n a_i e_i \mapsto \sum_1^n a_i \delta_i$, and let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B}$.

a) Show that $\phi_{V,B} : V \to V^*$ is an isomorphism, but that it depends on the choice of B.

b) Show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism that is independent of the choice of B (so we may denote it by ψ_V). For $v \in V$, show that $\psi_V(v)$ is the element of V^{**} taking $f \in V^*$ to $f(v)$.

c) Show that the association $V \mapsto \psi_V$ is natural in the following sense: If $F : V \to W$ is a vector space homomorphism with induced homomorphism $F^{**} : V^{**} \to W^{**}$ (notation as in problem 4), then $\psi_W \circ F = F^{**} \circ \psi_V$.