Math 602 Problem Set #9 Due Fri., Nov. 13, 2015

1. Let $p > 2$ be a prime number, and let $f(x) = x^{\frac{p-1}{2}} - 1$.

a) Show that every square in $(\mathbb{Z}/p)^\times$ is a root of $f(x) \in (\mathbb{Z}/p)[x]$.

b) Deduce that $f(x) = \prod_{i=1}^r (x - a_i)$, where f is as in (a) and where $\{a_1, \ldots, a_r\}$ is the set of squares in $(\mathbb{Z}/p)^\times$. [Hint: See PS 8 problem 7.]

c) Show that -1 is a square in $(\mathbb{Z}/p)^\times$ if and only if $p \equiv 1 (\mathrm{mod}\, 4)$. [Hint: $f(-1) = ?$]

2. a) Let $\alpha \in \mathbb{Z}[i]$. Show that if its norm $N(\alpha)$ is prime in $\mathbb{Z}$ then α is prime in $\mathbb{Z}[i]$.

b) Show that the converse fails. [Hint: PS 8, problem 8.]

3. Let $p > 0$ be a prime number in $\mathbb{Z}$.

a) Show that if $p \equiv 1 \,(\mathrm{mod}\, 4)$ then p is not prime in $\mathbb{Z}[i]$, but instead splits as the product of two distinct primes. [Hint: By problem 1(c), $p|(a^2+1)$ for some a; if p remained prime in $\mathbb{Z}[i]$ show $p|a \pm i$ and obtain a contradiction. For the second assertion, use norms.]

b) Show that if $p \equiv 3 \,(\mathrm{mod}\, 4)$ then p remains prime in $\mathbb{Z}[i]$. [Hint: If $p = \alpha\beta$, then $N(\alpha) = p$. Can p be the sum of two squares?]

c) Show that if $p = 2$ then up to a unit, p is the square of a prime in $\mathbb{Z}[i]$.

4. Suppose α is prime in $\mathbb{Z}[i]$, and let $p_1 \cdots p_r$ be the prime factorization of $N(\alpha)$ in $\mathbb{Z}$.

a) Show that $\alpha | p_j$ for some j.

b) Deduce that if $\alpha \notin \mathbb{Z} \cup i\mathbb{Z}$, then $N(\alpha)$ is prime. [Hint: Show $p_j = \alpha\beta$ with neither factor a unit, and then take norms.]

5. Show that $\alpha \in \mathbb{Z}[i]$ is prime if and only if either

(i) $\alpha = \varepsilon p$ where $\varepsilon \in \{\pm 1, \pm i\}$ and $p > 0$ is a prime in $\mathbb{Z}$ with $p \equiv 3 \,(\mathrm{mod}\, 4)$; or

(ii) $N(\alpha)$ is prime in $\mathbb{Z}$.

[Hint: Use problems 3 and 4.] Compare with your computations in PS 7 problem 8.

6. Let R be a commutative ring.

a) Let $I_1, \ldots, I_n$ be ideals in R, and let $\mathfrak{p} \subset R$ be a prime ideal containing $I_1 \cap \cdots \cap I_n$. Show that $I_i \subseteq \mathfrak{p}$ for some i.

b) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals in R, and let $I \subset R$ be an ideal that is contained in $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. Show that $I \subseteq \mathfrak{p}_i$ for some i. [Hint: Induction on n.]

c) Explain the content of (a) and (b) geometrically.

7. Let $I_1, \ldots, I_n \subset R$ be ideals in a commutative ring R.

a) Show that $\prod_{j=1}^n I_j = \bigcap_{j=1}^n I_j$ if the ideals are pairwise relatively prime. Explain this assertion geometrically in the case $R = \mathbb{C}[x, y]$.

b) Let $\phi : R \to \prod_{j=1}^n R/I_j$ be the map obtained by reducing modulo each I_j. Must this map be injective? surjective? an isomorphism? Give examples to show the possibilities, and prove a necessary and sufficient condition for ϕ to be an isomorphism.

8. a) Is the Jacobson radical always a radical ideal? Is the nilradical?

b) In each of the following rings, find the Jacobson radical, the nilradical, and the set of units. Also determine if the ring is local. $\mathbb{R}[x, y]$, $\mathbb{R}[[x, y]]$, $\mathbb{R}[x, y]/(y^3)$, $\mathbb{R}[x, y]/(xy)$, $\mathbb{R}[x, y]/(xy, y^3)$, $\mathbb{R}[x][[y]]$, $\mathbb{R}[[y]][x]$.

c) Prove that $\mathbb{R}[x]_{(x)} \subset \mathbb{R}[[x]]$, $\mathbb{R}[x, y]_{(x,y)} \subset \mathbb{R}[[x, y]]$, and $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$, but that $\mathbb{R}[x, y]_{(y)}$ is not a subring of $\mathbb{R}[x][[y]]$. Explain.