Math 602 Problem Set #7 Due Fri., Oct. 30, 2015

1. Which of the following are rings? (Note: To be a ring, it must have a multiplicative identity.) For those which are not, why not? For those which are, are they commutative? integral domains? fields?

a) $\mathbb{Q}[e^{2\pi i/15}]$.

b) $M_n(\mathbb{Z}[[x]])$.

c) $\mathbb{Z} \times \mathbb{R}$, with $(a,b)+(c,d)=(a+c,b+d)$, $(a,b)\cdot(c,d)=(ac,bd)$.

d) $\mathbb{R} \times \mathbb{R}$, with $(a,b)+(c,d)=(a+c,b+d)$, $(a,b)\cdot(c,d)=(ac-bd,ad+bc)$.

e) $\mathbb{Q}((x)) = \{\sum_{i=-n}^{\infty} a_i x^i \mid n \in \mathbb{Z}, a_i \in \mathbb{Q}\}$, the Laurent series in x over $\mathbb{Q}$.

f) $x\mathbb{R}[x]$, the polynomials that are multiples of x.

g) $\mathbb{H}_{\mathbb{Z}} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{Z}\}$, the quaternionic integers, under coordinate-wise addition, and with multiplication as in the quaternion group Q.

h) {continuous functions $f : \mathbb{R} \to \mathbb{R}$} under addition and multiplication.

i) {continuous functions $f : \mathbb{R} \to \mathbb{R}$} under addition and composition.

j) {continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support} (i.e. functions that vanish except in some interval $[-n,n]$), under addition and multiplication.

2. Which of the following are ring homomorphisms? For those which are not, why not? For those which are, what are the kernels and images?

a) $f : \mathbb{R}[x] \to \mathbb{R}$, $f(\sum a_i x^i) = \sum a_i 3^i \quad (a_i \in \mathbb{R})$

b) $f : \mathbb{C} \to \mathbb{C}$, $f(a+bi) = a-bi \quad (a,b \in \mathbb{R})$

c) $f : \mathbb{C} \to \mathbb{C}$, $f(a+bi) = a \quad (a,b \in \mathbb{R})$

d) $f : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}[\sqrt{3}]$, $f(a+b\sqrt{2}) = a+b\sqrt{3} \quad (a,b \in \mathbb{Z})$

e) $f : \mathbb{Z}[x] \to \mathbb{Q}$, $f(\sum a_i x^i) = \sum a_i/2^i \quad (a \in \mathbb{Z})$

f) $f : \mathbb{Z}[i] \to \mathbb{Z}/n$, $f(a+bi) = a+8b$. (Hint: Your answer should depend on n.)

3. Let $\mathbb{H} = \{$quaternions $a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$.

a) For $\alpha = a+bi+cj+dk \in \mathbb{H}$, define the conjugate $\bar{\alpha} = a-bi-cj-dk$, and define the absolute value $|\alpha| = \sqrt{a^2+b^2+c^2+d^2} \in \mathbb{R}_+$. Show that $|\alpha|^2 = \alpha\bar{\alpha}$ and that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$. Conclude that $|\alpha\beta| = |\alpha||\beta|$. Also, find all α such that $|\alpha| = 0$.

b) Does $\mathbb{H}$ have any (non-zero) zero-divisors? (Hint: Use part (a).)

c) Is $\mathbb{H}$ a division ring? a field?

4. Let $R_1 = \mathbb{R}[[x]] =$ the ring of formal power series in x, and let $R_2 = \mathbb{R}\{x\} =$ the ring of convergent power series in x (i.e. power series with a non-zero radius of convergence).

a) Show that R_1 and R_2 are integral domains. [Hint: For $f = \sum_{i=n}^{\infty} a_i x^i$ (with $n \geq 0$ and $a_n \neq 0$), define the *order* of f to be the integer $\mathrm{ord}(f) = n$. Define $\mathrm{ord}(0) = \infty$. What is $\mathrm{ord}(f \cdot g)$?]

b) Find all units in R_i.

c) Show that the set of non-units is a principal ideal (f) of R_i, for some $f \in R_i$. Show also that $R_i[1/f]$ is a field, and describe this field explicitly.

5. Show that $M_n(D)$ has no non-trivial two-sided ideals, for any division ring D.

6. Find an example of a non-zero map $f : \mathbb{Z} \to \mathbb{Z}/n$ (for some n) such that $f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)f(b)$ for all $a,b \in \mathbb{Z}$, but $f(1) \neq 1$ (and so f is not a homomorphism of rings under our definition.)