1. Find two extensions G of a fixed finite group B by a fixed finite abelian group A such that the two groups G are isomorphic as groups, but such that the two extensions $1 \to A \to G \to B \to 1$ are not isomorphic as extensions of B by A. [Hint: Try $A = C_3^2$ and $B = C_2$.]

2. Show that there is a unique group action of $\mathbb{Z}/2$ on $\mathbb{Z}/2$. With respect to that action, directly compute the groups $C^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $Z^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $B^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $H^2(\mathbb{Z}/2, \mathbb{Z}/2)$. In the case of H^2, interpret each element in terms of an extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$.

3. With respect to each of the actions of $\mathbb{Z}/2$ on $\mathbb{Z}/3$, compute $H^0(\mathbb{Z}/2, \mathbb{Z}/3)$, $H^1(\mathbb{Z}/2, \mathbb{Z}/3)$, $H^2(\mathbb{Z}/2, \mathbb{Z}/3)$. How does each H^2 relate to group extensions?

4. Let G be a finite group and let p be a prime number. Show that G contains a subgroup F of order prime to p such that for every quotient $E := G/N$ of G of order prime to p, the composition $F \hookrightarrow G \to E$ is surjective. Do this in steps as follows:

i) Let $Q \subseteq G$ be the subgroup generated by all the Sylow p-subgroups of G. Let P be a Sylow p-subgroup of G, and let $G' = N_G(P)$. Show that Q is a normal subgroup of G, and that the quotient map $\pi : G \to H := G/Q$ restricts to a surjection $\pi' : G' \to H$. [Hint: Say $\pi(g) = h$. Must P and gPg^{-1} be conjugate subgroups of Q? Does this yield an element of G' that maps to h?] Show that H is the largest quotient of G of order prime to p.

ii) Deduce that G' (and hence also G) contains a subgroup F having order prime to p such that $\pi(F) = H$, and that F has the desired property. [Hint: With $Q' = N_Q(P)$, consider the exact sequences $1 \to Q' \to G' \to H \to 1$, $1 \to P \to G' \to G'/P \to 1$, and $1 \to Q'/P \to G'/P \to H \to 1$, and apply Schur-Zassenhaus to one of them.]

5. Let $0 \to A \xrightarrow{i} G \xrightarrow{\pi} B \to 1$ be a short exact sequence of finite groups, with A abelian (written additively). For each $b \in B$ pick some $g_b \in G$ such that $\pi(g_b) = b$. Define an action α of B on A by $b \cdot a = g_b a g_b^{-1}$. For $b_1, b_2 \in B$, define $f(b_1, b_2) \in A$ by $g_{b_1} g_{b_2} = f(b_1, b_2) g_{b_1 b_2}$.

a) Verify that $f \in Z^2_\alpha(B, A)$; i.e. that $f(b_1, b_2) + f(b_1 b_2, b_3) = b_1 \cdot f(b_2, b_3) + f(b_1, b_2 b_3)$. [Hint: Evaluate $g_{b_1} g_{b_2} g_{b_3}$ in two ways.]

b) Verify that $(a_1 g_{b_1})(a_2 g_{b_2}) = (a_1 + b_1 \cdot a_2 + f(b_1, b_2)) g_{b_1 b_2} \in G$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, giving the multiplication law in G.

c) Suppose that for each $b \in B$ we have another choice $g'_b \in G$ of an element in G with $\pi(g'_b) = b$, and let f' be the analogous element of $Z^2_\alpha(B, A)$. For each $b \in B$ define $e(b) \in A$ by $g'_b = e(b) g_b$. Verify that $f'(b_1, b_2) - f(b_1, b_2) = e(b_1) + b_1 \cdot e(b_2) - e(b_1 b_2)$; i.e. f, f' differ by an element of $B^2_\alpha(B, A)$. [Hint: Evaluate $g'_{b_1} g'_{b_2}$ in two ways.]