Math 602 Problem Set #5 Due Wed., Oct. 14, 2015

1. Suppose that $N \triangleleft S_5$.

a) Show that if N contains a transposition (a, b) then $N = S_5$. (Hint: The set of transpositions generates S_5.)

b) Show that if $N \cap A_5 = 1$ and $\sigma \in N$, then either $\sigma = 1$ or else σ is a transposition. (Hint: Show that $\sigma^2 = 1$.)

c) Conclude that $N = 1, A_5$, or S_5.

2. Let $n > 2$. Show that if the dihedral group D_n of order $2n$ is isomorphic to a semi-direct product $C_r \rtimes C_s$, then $r = n$ and $s = 2$.

3. Verify the properties of an internal semi-direct product. More precisely:

Consider a short exact sequence $1 \to N \to G \to H \to 1$ of groups, and a section $s : H \to G$. Show that H is isomorphic to $s(H)$; that G is generated by its subgroups N and $s(H)$; and that $N \cap s(H) = 1$. Show that $s(H)$ need not be normal in G, although N must be normal. Also show that there is a well-defined action of H on N, defined by the conjugation action of $s(H)$.

4. Show that the construction of an external semi-direct product $N \rtimes H$ works. More precisely:

Let α be an action of a group H on a group N; let G be the set $N \times H$; and define a product law on G by

$$(n, h)(n', h') = (n(h \cdot n'), hh'),$$

where $h \cdot n'$ indicates the action given by α. Show that under this product law, G is a group, containing a normal subgroup $\bar{N}$ that is isomorphic to N and also containing a subgroup $\bar{H}$ that is isomorphic to H, such that: G is generated by $\bar{N}$ and $\bar{H}$; $\bar{N} \cap \bar{H} = 1$; and the conjugation action of $\bar{H}$ on $\bar{N}$ is given by α (with respect to your chosen isomorphisms $H \cong \bar{H}$ and $N \cong \bar{N}$).

5. Which of the following groups are isomorphic: $C_2 \wr C_2$, Q, D_4, C_2^3?

6. Find all groups of order 66, up to isomorphism. Which are simple? solvable? nilpotent? abelian? cyclic? Which are split extensions (of a non-trivial quotient by a non-trivial subgroup)?

7. a) Show directly that every group of order 56 is solvable. [Hint: How many elements have order 7?]

b) Consider the finite groups whose order is 56 and whose exponent is 14. For each such group, let N_p be the number of Sylow p-subgroups, for $p = 2, 7$.

(i) Do there exist such groups with $N_2 = N_7 = 1$?

(ii) Do there exist such groups with $N_7 = 1$ and $N_2 > 1$?

(iii) Do there exist such groups with $N_2 = 1$ and $N_7 > 1$?

(iv) Do there exist such groups with $N_2 > 1$ and $N_7 > 1$?