Math 602 Problem Set #2 Due Mon., Sept. 21, 2015

1. If N is a subgroup of G, call N a *characteristic* subgroup if N is invariant under all automorphisms of G; i.e. if $\phi(N) = N$ for every $\phi \in \mathrm{Aut}(G)$.

a) Show that every characteristic subgroup of a group G is normal in G, but that the converse is not in general true.

b) Is the commutator subgroup of G (i.e. the subgroup generated by the commutators $[a, b] := aba^{-1}b^{-1}$) characteristic?

c) Is the Frattini subgroup of G (cf. PS#1, problem 4) characteristic?

d) Is the subgroup N, generated by the squares in G and the elements in G of order 17, characteristic?

e) Is every subgroup of index 2 characteristic?

2. Consider the following sentence: "If M is a __________ subgroup of N, and N is a __________ subgroup of G, then M is a __________ subgroup of G." Of the eight ways of filling in the blanks with "normal" or "characteristic", find which are true. Prove these, and find counterexamples for the others.

3. Find all normal subgroups of the quaternion group Q. Find the Frattini subgroup Φ, the center Z, and the commutator subgroup Q'. What is Q/Φ? Q/Z? Q/Q'?

4. If S is a subset of a group G, say that S *generates* G if the subgroup of G generated by S is all of G (i.e. no proper subgroup of G contains S). Call $g \in G$ a *non-generator* of G if whenever a subset $S \subset G$ doesn't generate G, neither does $S \cup \{g\}$. Say G is finite.

a) Show that the Frattini subgroup $\Phi \subseteq G$ is the set of non-generators of G.

b) Show that if S is a subset of G, and $\bar{S}$ is the image of S under $G \to G/\Phi$, then S generates G if and only if $\bar{S}$ generates G/Φ.

5. Consider abelian groups A_i, for some set of consecutive integers i, along with homomorphisms $f_i : A_i \to A_{i+1}$. We say that the sequence $\cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots$ is a *complex* if the composition $f_i f_{i-1}$ is zero for all i, and that it is *exact* if $\ker(f_i) = \mathrm{im}(f_{i-1})$ for each i.

a) Show that every exact sequence is a complex. Give an example of a complex that is not an exact sequence.

b) Show that $A \to B$ is injective if and only if there is an exact sequence $0 \to A \to B \to C \to 0$ that includes this map. Show that $B \to C$ is surjective if and only if there is an exact sequence $0 \to A \to B \to C \to 0$ that includes this map.

c) What does it mean to say that $0 \to A_1 \to A_2 \to 0$ is exact? What does it mean to say that $0 \to A \to B \to C \to 0$ is exact?

d) Show that if $A \to B \to C \to 0$ and $0 \to C \to D \to E$ are exact, then so is $A \to B \to D \to E$, where the map $B \to D$ is the composition of the given maps $B \to C$ and $C \to D$. (This says that exact sequences can be "spliced".)

e) Do parts (a)-(d) continue to hold if we consider vector spaces over a given field F, rather than considering abelian groups? Do some parts still hold but not others? What if we instead consider groups that need not be abelian? What if we instead consider commutative rings with identity? What if we consider pointed sets (i.e. sets together with a distinguished element "0"), with maps that are required to take 0 to 0?