Quadratic Forms (Math 520/620/702)

Problem Set #5 Due Wed., Dec. 7, 2011, in class.

1. a) Which of the following elements lie in $\mathbb{F}_3((t))$? For each one that does, find this element explicitly, i.e. in the form $\sum_{i=n}^{\infty} a_i t^i$, where $n \in \mathbb{Z}$ and each $a_i \in \mathbb{F}_3$.

$$\frac{1+t^3}{t^2},\ \frac{1}{1-t},\ \sqrt{1+t},\ \sqrt{t},\ \sqrt{2+t}$$

b) Which of the following elements lie in $\mathbb{Q}_3$? For each one that does, find this element explicitly, i.e. in the form $\sum_{i=n}^{\infty} a_i 3^i$, where $n \in \mathbb{Z}$ and each $a_i \in \mathbb{Z}$. If you can, write the element in a form so that each $a_i \in \{0,1,2\}$.

$$\frac{41}{9},\ -\frac{1}{2},\ \frac{1}{2},\ \sqrt{7},\ \sqrt{2},\ \sqrt{3}$$

2. a) Prove the product formula for absolute values on $\mathbb{Q}$. That is, show that if $a \neq 0$, then

$$\prod_{v \in \Omega} |a|_v = 1.$$

(Here, $\Omega = \{\infty, 2, 3, 5, \ldots\}$ indexes the equivalence classes of non-trivial absolute values on $\mathbb{Q}$.)

b) State and prove the analogous statement for $\mathbb{F}_p(t)$.

3. a) Prove that for every positive integer N there is an integer n such that

$$n^3 - 5n \equiv 3 \pmod{5^N}.$$

[Hint: Use $\mathbb{Q}_5$.]

b) Find such an n if $N = 3$.

[Hint: Use Newton's Method: $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \pmod{5^{i+1}}$.]

4. a) Find all anisotropic quadratic forms over $\mathbb{F}_3((t))$ up to equivalence.

b) Which of these forms remain anisotropic over $\mathbb{F}_9((t))$? Which of them become isometric to each other over $\mathbb{F}_9((t))$?

c) Redo part (b) with $\mathbb{F}_3((\sqrt{t}))$ instead of $\mathbb{F}_9((t))$.

5. a) Show that the Hilbert symbol $(\ ,\ )_p := (\ ,\ )_{\mathbb{Q}_p}$ is symmetric, bimultiplicative (i.e. $(a, bc)_p = (a,b)_p(a,c)_p$, and similarly with the entries reversed), and satisfies $(a, 1-a)_p = 1$ if $a, 1-a \in \mathbb{Q}_p^\times$.

b) For each odd prime number p, determine whether there is an element $z_p \in \mathbb{Q}_p^\times$ such $(-1, z_p)_p = -1$. If there is, find such a z_p explicitly.

6. Consider the following quadratic forms over $\mathbb{Q}$:

$$q_1 = \langle 1,1,1,-1\rangle,\ q_2 = \langle 1,1,1,-7\rangle,\ q_3 = \langle 1,1,1,1,1\rangle.$$

a) For each i, determine whether the form q_i is isotropic over $\mathbb{Q}$.

b) For each i, find all the completions of $\mathbb{Q}$ (i.e. $\mathbb{Q}_p$ or $\mathbb{R}$) over which q_i is isotropic. Does your answer, in conjunction with your answer to part (a), agree with Hasse-Minkowski?

7. Let q be a quadratic form over a global field F, and let Ω be the set of (equivalence classes of) non-trivial absolute values on F.

a) Prove that q is hyperbolic over F if and only if it is hyperbolic over F_v for all $v \in \Omega$.

b) Let i_W denote the Witt index of a quadratic form, and for $v \in \Omega$ let q_v denote q viewed as a form over F_v. Prove that $i_W(q) = \min_{v \in \Omega} i_W(q_v)$.

8. From the exercises in Lam, Chapter VI (pages 183-186), do problems 5-7. [Hint: See Examples 2.32-2.34 in Chapter VI.]