Quadratic Forms (Math 520/620/702)

Problem Set #4 Due Wed., Nov. 23, 2011, in class.

1. Let A be a central simple algebra over F, and let E be a field that contains F and is contained in A.

a) Show that the centralizer $C_A(E)$ contains E, and is an E-algebra.

b) Show that $\dim_F(C_A(E)) = \dim_E(C_A(E))[E : F]$.

c) Deduce that $[E : F]$ divides the degree of the F-algebra A, with equality if and only if $C_A(E) = E$. [Hint: What is $\dim_F(E) \cdot \dim_F(C_A(E))$?]

d) Show that if $[E : F]$ is equal to the degree of A, then E is a *maximal subfield* of A (i.e. E is not strictly contained in any other field E' with $F \subseteq E' \subseteq A$).

e) Show that if A is a division algebra over F then the converse of (d) holds. [Hint: If not, show there exists $a \in C_A(E)$ that does not lie in E, and consider $E(a) \subseteq A$.]

2. a) Let D be a non-commutative division ring that is also a finite dimensional $\mathbb{R}$-algebra. Show that the center must be $\mathbb{R}$, and hence D is a (central) division algebra over $\mathbb{R}$. [Hint: If not, D is a non-trivial central simple algebra over the field $Z(D)$. What can that field be?]

b) Let E be a maximal subfield of the $\mathbb{R}$-division algebra D. Show that $E \cong \mathbb{C}$ and that the degree of D over $\mathbb{R}$ is 2. Deduce that D is a quaternion algebra over $\mathbb{R}$.

c) Conclude that $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

3. Let K and A be as in problem 6 of Problem Set 3, and preserve the notation from that problem.

a) Show that if A is a K-division algebra then A contains a maximal subfield E of degree n over K such that E is a cyclic Galois extension of K, i.e. a Galois extension of K whose Galois group is cyclic. (For this reason, A is referred to as a *cyclic algebra*.) Find the centralizer of E in A.

b) Show that if $b = 1$ then A is isomorphic to a matrix algebra over K. [Hint: Consider the matrices M, N.] What does this say if $n = 2$?

c) Given an example to show that A is not always isomorphic to a matrix algebra.

4. If σ is a permutation of $\{1, 2, 3, 4\}$, consider the map $f_\sigma : \mathbb{H} \to \mathbb{H}$ that takes $a_1 + a_2 i + a_3 j + a_4 k$ to $a_{\sigma(1)} + a_{\sigma(2)} i + a_{\sigma(3)} j + a_{\sigma(4)} k$, where each $a_i \in \mathbb{R}$.

a) For which permutations σ is f_σ an automorphism of $\mathbb{H}$?

b) Concerning each such σ, what assertion does the Skolem-Noether Theorem make?

c) Verify this assertion explicitly by finding an element as asserted in that theorem, for one such choice of σ (other than the identity).

5. Do the following problems from Lam, Chapter V (pages 140-142):

a) Exercise 4.

b) Exercise 12.

c) Exercise 14.