Quadratic Forms (Math 520/620/702)

Problem Set #2 Due Mon., Oct. 24, 2011, in class.

1. Let $F = \mathbb{C}((t))$.

a) Show that $F^\times/F^{\times 2}$ has exactly two elements, represented by $\{1, t\}$. [Hint: Show that if $f \in \mathbb{C}[[t]]$ has a non-zero constant term, then f is a square.]

b) Show that every binary quadratic form over F is universal. [Hint: Use part (a) to reduce to just a few possibilities.]

c) Deduce that $u(F) = 2$.

d) Describe the structure of $W(F)$, $Q(F)$, $I(F)$, and $I^2(F)$.

2. Let k be an arbitrary field of characteristic unequal to 2. Let $F = k((t))$ and $R = k[[t]]$.

a) Let f be an element of R with constant term c. Show that f is a unit in the ring R if and only if $c \neq 0$. Also show that if $c = 1$ then f is a square in R. [Hint: Taylor series for $(1+x)^{1/2}$.] Deduce that if $c \neq 0$, then f is a square in R (and in F) if and only if $c \in k^{\times 2}$.

b) Let $q = \langle a_1, \ldots, a_n \rangle$ with $a_i \in R^\times$, the group of units in R. Show that if q is isotropic over F then $q(x) = 0$ for some $x = (x_1, \ldots, x_n) \in R^n$ that does not lie in tR^n.

c) In (b), write $a_i = c_{i,0} + c_{i,1}t + c_{i,2}t^2 + \cdots$ with $c_{i,j} \in k$, and write $\bar{q} = \langle c_{1,0}, \ldots, c_{n,0} \rangle$. Show that if q is isotropic over F then $\bar{q}$ is isotropic over k. [Hint: Use part (b) and then reduce mod (t).]

d) Prove the converse of part (c). [Hint: Use part (a).]

3. In this problem, we retain the notation of problem 2.

a) Show that every regular quadratic form over F is equivalent to a quadratic form $q_1 \perp tq_2$ for some $q_1 = \langle a_1, \ldots, a_r \rangle$ and some $q_2 = \langle a_{r+1}, \ldots, a_n \rangle$, where each $a_i \in R^\times$. Show moreover that if $\bar{q}_1$ or $\bar{q}_2$ is isotropic, then so is q. [Hint: Use problem 2(d).]

b) Prove the converse of the last part of (a). [Hint: First obtain an $x \in R^n$ as in problem 2(b). Next, consider the case in which at least one of the elements $x_1, \ldots, x_r \in R$ has non-zero constant term; and handle this case by modding out by (t). Finally, handle the remaining case by showing that the form $t^2 q_1 + tq_2$ is also isotropic over R, and then dividing by t and reducing to the previous case.]

c) Using parts (a) and (b), find and prove a formula that relates $u(F)$ to $u(k)$.

4. Let F be a finite field whose order is congruent to 3 modulo 4. Show directly that $\langle -1, -1 \rangle = \langle 1, 1 \rangle$ in $\widehat{W}(F)$ and hence in $W(F)$; that $\langle 1, 1, 1 \rangle = \langle -1 \rangle$ in $W(F)$; and that $\langle 1, 1, 1, 1 \rangle$ is trivial in $W(F)$. (Here, "directly" means that you should work with these quadratic forms, and not just use that we showed that $W(F)$ is abstractly isomorphic to $\mathbb{Z}/4$.)

5. a) Verify the assertion in Lam (page 34 of the text, and also exercise 18 on page 49) that the subfield $\bigcup_{n \geq 1} \mathbb{F}_5(\sqrt[2^n]{2})$ of $\bar{\mathbb{F}}_5$ is quadratically closed (and hence is the quadratic closure of $\mathbb{F}_5$ in $\bar{\mathbb{F}}_5$).

b) Find a similar description for the quadratic closure of $\mathbb{F}_3$ in $\bar{\mathbb{F}}_3$.

c) Is there a similar simple description for the quadratic closure of $\mathbb{Q}$? Why or why not?

6. Do the following problems from Lam, Chapter II (pages 47-49):

a) Exercise 2. [Hint: Use another form of equivalence.]

b) Exercise 5. [Hint: Proceed similarly to Theorem 3.5.]

c) Exercise 6(a). [Hint: See the example on page 35.]

d) Exercise 7(a). [Hint: First do the case $n = 2$ (and even $n = 1$), and guess a pattern.]

e) Exercise 10. [Hint: Reinterpret the condition on -1 in terms of isotropy.]

f) Exercise 11(1). [Hint: Use the explicit group isomorphism $Q(F) \to W(F)/I^2(F)$ to reinterpret the multiplication on $W(F)/I^2(F)$ as an operation $\circ$ on $Q(F)$.