Quadratic Forms (Math 520/620/702)

Problem Set #1 Due Wed., Oct. 5, 2011, in class.

1. a) Let (V, q) be a quadratic space, and assume that $V = V_1 \textcircled{\perp} V_2$ for some subspaces $V_1, V_2 \subseteq V$. Show that $\mathrm{rad}(V) = \mathrm{rad}(V_1) \textcircled{\perp} \mathrm{rad}(V_2)$, and deduce that V is regular if and only if V_1 and V_2 are both regular.

b) What if instead we just assume $V = V_1 \oplus V_2$?

2. a) Let r be a positive integer and let F be a field (as usual, of characteristic $\neq 2$). Show that the following two conditions are equivalent:

(i) Every regular quadratic form in r variables over F is universal.

(ii) Every regular quadratic form in $r + 1$ variables over F is isotropic.

b) Find the values of r for which these equivalent conditions hold, if $F = \mathbb{C}$. Do the same if $F = \mathbb{F}_3$ (the field of three elements).

3. Let q be a regular quadratic form.

a) Prove that there is an anisotropic vector for q.

b) Prove that every element of $\mathrm{O}(q)$ has determinant ± 1.

c) Prove that $\mathrm{SO}(q)$ is a normal subgroup of index two in $\mathrm{O}(q)$.

d) Do parts (a) - (c) still hold if q is not assumed regular?

4. a) Let (V, q) be a regular quadratic space, with subspaces X, Y. Show that if $\sigma : X \to Y$ is an isometry, then there is an isometry $\tau : V \to V$ whose restriction to X is σ. (Hint: Use Witt Cancellation.)

b) Deduce that if x, y are non-zero vectors in V, and if $q(x) = q(y)$, then there is an isometry $\tau : V \to V$ such that $\tau(x) = y$ (even if $q(x) = 0$). Where does your proof fail if we instead permit x to be equal to 0?

5. Let q be a regular quadratic form on $V = F^n$, corresponding to the invertible $n \times n$ symmetric matrix M. For $A \in \mathrm{M}_n(F)$, define $\tau(A) = M^{-1}A^{\mathrm{t}}M$, where A^{t} is the transpose of A.

a) Show that τ is an *involution* on $\mathrm{M}_n(F)$, in the sense that $\tau(A+B) = \tau(A)+\tau(B)$; τ^2 is the identity; and $\tau(AB) = \tau(B)\tau(A)$.

b) Show that

$$\mathrm{O}(q) = \{A \in \mathrm{GL}_n(F) \mid \tau(A) = A^{-1}\},$$

and deduce that

$$\mathrm{SO}(q) = \{A \in \mathrm{SL}_n(F) \mid \tau(A) = A^{-1}\}.$$

Explain why these equalities are familiar in the case that $q = \sum x_i^2$.

c) Let

$$\mathrm{Skew}(q) = \{A \in \mathrm{M}_n(F) \mid \tau(A) = -A\}.$$

What is $\mathrm{Skew}(q)$ if $q = \sum x_i^2$?

d) View $\mathrm{SO}(q)$ and $\mathrm{Skew}(q)$ as subsets of the F-vector space $\mathrm{M}_n(F) \cong F^{n^2}$. Show that each can be defined as the locus of common zeroes of a set of polynomials in n^2 variables. (Such polynomial loci are called *algebraic varieties*.) Is either one a vector space?

(continued)

6. Retain the notation of problem 5.

a) Suppose that $A \in \mathrm{GL}_n(F)$ and that -1 is not an eigenvalue of A. Show that $A+I$ is invertible (where I is the identity).

b) If A is as in part (a), let $B = (A+I)^{-1}(A-I)$. Show that 1 is not an eigenvalue of B and that $I-B$ is invertible. Show also that $A = (I+B)(I-B)^{-1}$.

c) In the situation of part (b), show that if $A \in \mathrm{SO}(q)$ then $B \in \mathrm{Skew}(q)$, and conversely.

d) Let

$$\mathrm{SO}(q)^\circ = \{A \in \mathrm{SO}(q) \mid A+I \in \mathrm{GL}_n(F)\}$$

and let

$$\mathrm{Skew}(q)^\circ = \{A \in \mathrm{Skew}(q) \mid I-B \in \mathrm{GL}_n(F)\}.$$

Show that the association $A \mapsto B$ as above defines a bijection $C : \mathrm{O}(q)^\circ \to \mathrm{Skew}(q)^\circ$. Show moreover that the maps C and C^{-1} are defined by systems of polynomials in n^2 variables. (Such a bijection is called an *isomorphism of varieties.*)

e) Show that there is a polynomial in n^2 variables that does not vanish identically on $\mathrm{SO}(q)$ and such that $\mathrm{SO}(q)^\circ$ is the complement in $\mathrm{SO}(q)$ of the zero locus of this polynomial. Do the same for $\mathrm{Skew}(q)^\circ$ in $\mathrm{Skew}(q)$.

f) Deduce that a dense open subset of $\mathrm{SO}(q)$ is isomorphic to a dense open subset of some vector space. In algebraic geometry, one then says that $\mathrm{SO}(q)$ is a *rational variety.* (If you know about the Zariski topology, then you can interpret this problem for any field F of characteristic not 2. Otherwise, you can restrict to the case that $F = \mathbb{R}$ or $\mathbb{C}$.)