Math 503 Problem Set #8 Due the week of March 12, 2018, in lab.

Read Artin, Chapter 13, sections 5-7 (optional: 8-10); and Chapter 14, sections 1-4.

1. a) From Artin, Chapter 13, do these problems (pages 408-411): 5.1(a), 6.3, 7.1.
b) From Artin, Chapter 14, do these problems (pages 437-441): 1.1, 2.1.

2. a) Which of the following rings are integrally closed (in their fraction field)? $\mathbb{R}[x]$, $\mathbb{R}[x,y]/(y^2-x)$, $\mathbb{R}[x,y]/(y^2-x^3)$, $\mathbb{R}[x,y]/(y^2-x^2-x^3)$.

b) For each ring in (a), draw the corresponding variety on which it is the ring of functions. Any conjectures about the relationship between the geometry of the curve and whether the ring is integrally closed?

3. a) If ℓ is a prime, for which positive integers n is there a primitive n^{th} root of unity in $\mathbb{F}_\ell$? [Hint: What is the structure of the group $\mathbb{F}_\ell^\times$?]

b) Given a prime number p, for which primes ℓ does the cyclotomic polynomial $\Phi_p(x)$ factor in $\mathbb{F}_\ell[x]$ as a product of distinct monic linear factors? [Hint: Use part (a).]

4. a) Show that $R = \mathbb{Z}[\sqrt{2}]$ is a Euclidean domain. [Hint: Consider the lattice in Figure 13.9.6 in Artin, and for $\alpha \in R$ let $\sigma(\alpha)$ be the square of the distance from the origin to the lattice point corresponding to α.] Is R a PID? a UFD?

b) Show that a prime number $p > 0$ factors non-trivially in R if and only if $\pm p$ is of the form $a^2 - 2b^2$ for some $a, b \in \mathbb{Z}$. [Hint: For $\alpha = a + b\sqrt{2} \in R$ (with $a, b \in \mathbb{Z}$), let $\overline{\alpha} = a - b\sqrt{2}$, and consider the norm $N(\alpha) := \alpha\overline{\alpha}$.]

c) Show that a prime number $p > 0$ remains prime in R if and only if there is no square root of 2 in $\mathbb{F}_p$. [Hint: Mimic the proof for $\mathbb{Z}[i]$.]

d) For $p = 3, 5, 7, 17$, determine whether p remains prime in R, and whether there exist integers a, b such that $a^2 - 2b^2 = \pm p$.

5. Which of the following $\mathbb{Z}$-modules are finitely generated? Which are free?

a) $M = \mathbb{Z}/5 \times \mathbb{Z}/7$
b) $M = (5) =$ ideal generated by 5
c) $M = \mathbb{Z}[\sqrt{3}]$
d) $M = \mathbb{Z}[\pi]$
e) $M = \mathbb{Z}[1/2]$
f) $M = \frac{1}{2}\mathbb{Z} = \{\frac{n}{2} \mid n \in \mathbb{Z}\}$