Read Artin, Chapter 11, sections 4-8.

1. From Artin, Chapter 11, do these problems (pages 354-358):
3.9, 3.12, 3.13, 7.1, 7.3.

2. Which of the following commutative rings are integral domains? Which are fields? $\mathbb{Z}/21\mathbb{Z}$, $\mathbb{R}[x]/(x^2+2)$, $\mathbb{R}[x]/(x^2-2)$, $\mathbb{F}_2[x]/(x^2-1)$, $\mathbb{F}_2[x]/(x^2-x-1)$, $\mathbb{Z}[x]/(2,x)$, $\mathbb{Z}[x]/(x-2)$, $\mathbb{R}[x,y]/(y-x^2)$, $\mathbb{R}[x,y]/(x,y-x^2)$, $\mathbb{R}[x,y]/(y,y-x^2)$, $\mathbb{R}[x]/(x^4+1)$.

3. Which of the following ideals are maximal in the indicated rings? For those that are not, find a maximal ideal containing the given ideal. Explain your assertions. [Caution: One of these is especially tricky.]
$(x-3) \subset \mathbb{Q}[x]$; $(x-3) \subset \mathbb{Z}[x]$; $(x^2-3) \subset \mathbb{R}[x]$; $(x^2+3) \subset \mathbb{R}[x]$; $(x-3) \subset \mathbb{C}[x,y]$; $(x^2+1, y-3) \subset \mathbb{R}[x,y]$; $(x^2+1, y-3) \subset \mathbb{C}[x,y]$; $(x^2+1, y^2+1) \subset \mathbb{R}[x,y]$.

4. Let R be a commutative ring and $f(x) \in R[x]$ a polynomial of degree $n > 0$.

a) Show that if R is an integral domain then $f(x)$ has at most n roots in R. [Hint: Use induction to show that if $a_1, \ldots, a_r \in R$ are distinct roots of $f(x)$, then the product $(x-a_1)\cdots(x-a_r)$ divides $f(x)$.]

b) Show by example that the same assertion need not hold if R is not an integral domain. [Hint: Try $R = \mathbb{Z}/8[x]$ and $f(x) = x^2 - c$ for some $c \in \mathbb{Z}/8$.] Where does your proof in part (a) break down in this situation?

5. If a, b, c are non-zero elements of a ring R, we say that $a = bc$ is a *non-trivial factorization* of a if neither b nor c is a unit in R.

a) Which elements of $\mathbb{Z}$ can be factored non-trivially? Which elements of $\mathbb{R}[x]$?

b) Find all the units in the ring $\mathbb{Z}[i]$ of Gaussian integers.

c) Which of the following elements of $\mathbb{Z}[i]$ can be factored non-trivially? For each one that can be, do so explicitly. $2, 3, 5, 7, 11, 13, 15, 3i, 5i, 2+i, 3+i$

d) Make a conjecture about which Gaussian integers can be factored non-trivially.