Math 502 Problem Set #14 Due week of Dec. 4, 2017, in lab.

Read Artin, Chapter 7, sections 9-10; Chapter 8, sections 4-6; Chapter 10, sections 1-8.

1. From Artin, Chapter 7, do problems 7.3, 10.5-10.7 (pages 221-228).

2. a) Find all groups of order 39, up to isomorphism.

   b) Find all groups of order 28, up to isomorphism.

In particular, for each of these two parts, determine how many isomorphism classes there are, of groups of the given order.

Note: In the following problems you may wish to use Theorem 10.4.6 in Artin.

3. Let G be a group of order n, with conjugacy classes $C_1, \ldots, C_r$, and $c_j = |C_j|$. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G, each viewed as a class function $\chi_i : \{C_1, \ldots, C_r\} \to \mathbb{C}$. Define the $r \times r$ matrix A by $a_{i,j} = \sqrt{\frac{c_j}{n}}\, \chi_i(C_j) \in \mathbb{C}$.

   a) Show that the rows of A are orthonormal vectors in $\mathbb{C}^r$ (under the standard Hermitian dot product).

   b) Deduce that A is a unitary matrix and that its columns are orthonormal vectors in $\mathbb{C}^r$.

   c) Conclude that if C, C' are conjugacy classes in G then $\sum_{i=1}^{r} \chi_i(C)\overline{\chi_i(C')}$ is equal to 0 if $C \neq C'$ and is equal to $n/|C|$ if $C = C'$. (This is the second orthogonality relation.) Explain why this determines χ_r if you know $\chi_1, \ldots, \chi_{r-1}$.

4. a) Show that D_4 has four one-dimensional representations, and find them explicitly.

   b) Show that there is exactly one more irreducible representation of D_4, and find its dimension.

   c) Explicitly find the characters of all five irreducible representations.

5. Consider the quaternion group of order eight (Artin, page 47).

   a) Determine how many irreducible representations this group has.

   b) For each of these irreducible representations, find their dimensions and their characters.