Math 502 Problem Set #11 Due week of Nov. 13, 2017, in lab.

Read Artin, Chapter 6, sections 2, 3, 8-11. Optional: sections 4-6, 12.

1. From Artin, Chapter 6, do problems 8.1, 8.2, 9.1, 9.3, 11.4 (pages 188-194).

2. a) Show that if $A \in \mathrm{GL}_n(\mathbb{R})$ then $Av \cdot w = v \cdot A^{\mathrm{t}} w$ for all $v, w \in \mathbb{R}^n$ (taking the usual dot product). What does this say if A is symmetric? if A is orthogonal?

b) Show that if $A \in \mathrm{GL}_n(\mathbb{C})$ then $Av \cdot w = v \cdot A^* w$ for all $v, w \in \mathbb{C}^n$ (taking the Hermitian dot product given by $v \cdot w = \sum_{j=1}^n \bar{a}_j b_j$ for $v = (a_1, \dots, a_n)$ and $w = (b_1, \dots, b_n)$, as in the textbook). What does this say if A is Hermitian? if A is unitary?

3. a) Let F be a field that is not of characteristic 2, and let B be a symmetric bilinear form on an F-vector space V. Let $q = q_B$ be the associated quadratic form (as in Problem Set #10, problem 4(b)). Let $T \in \mathrm{End}(V)$. Show that T preserves q (i.e., $q(T(v)) = q(v)$ for all $v \in V$) if and only if T preserves B (i.e., $T \in \mathrm{O}(V, B)$). [Hint: In one direction, use the formula in Problem Set #10, problem 5(a).]

b) Using part (a), deduce that $A \in \mathrm{GL}_n(\mathbb{R})$ preserves the Euclidean length (i.e., $|Av| = |v|$ for all $v \in \mathbb{R}^n$) if and only if $A \in \mathrm{O}_n$ (i.e., $Av \cdot Aw = v \cdot w$ for all $v, w \in \mathbb{R}^n$).

4. Consider the Hermitian dot product on $\mathbb{C}^n$, and for $v \in \mathbb{C}^n$ write $|v| = (v \cdot v)^{1/2}$ (non-negative square root).

a) Given $v = (a_1, \dots, a_n) \in \mathbb{C}^n$, write $a_j = b_j + ic_j$ with $b_j, c_j \in \mathbb{R}$, and let $\tilde{v} = (b_1, c_1, \dots, b_n, c_n) \in \mathbb{R}^{2n}$. Show that $|v| = |\tilde{v}|$, where the left hand side is the norm taken in $\mathbb{C}^n$ as above and the right hand side is the usual Euclidean norm taken in $\mathbb{R}^{2n}$.

b) Show that $v \cdot w = \frac{1}{4}\left(|v+w|^2 - |v-w|^2 + i|v-iw|^2 - i|v+iw|^2\right)$ for all $v, w \in \mathbb{C}^n$.

c) Let $A \in \mathrm{GL}_n(\mathbb{C})$. Use the formula in part (b) to show that A preserves the Euclidean length on $\mathbb{C}^n$ (viewed as $\mathbb{R}^{2n}$) if and only if $A \in \mathrm{U}_n$ (i.e., $Av \cdot Aw = v \cdot w$ for all $v, w \in \mathbb{C}^n$). [Hint: Proceed analogously to what was done in problem 3.]

5. a) Suppose that $A \in M_n(\mathbb{R})$ and that -1 is not an eigenvalue of A. Show that $A + I$ is invertible.

b) If A is as in part (a), let $B = (A+I)^{-1}(A-I)$. Show that 1 is not an eigenvalue of B and that $I - B$ is invertible. Show also that $A = (I+B)(I-B)^{-1}$.

c) Show that if A is an orthogonal matrix then B is a skew-symmetric matrix (i.e. $B^{\mathrm{t}} = -B$), and conversely.