Math 502 Problem Set #10 Due week of Nov. 6, 2017, in lab.

Read Artin, Chapter 5, sect. 1; and Chapter 8, sect. 1-3.

1. From Artin, Chapter 5, do problem 1.3 (page 150). From Artin, Chapter 8, do problems 1.1, 2.2, 3.3 (page 254).

2. Let V be a finite dimensional vector space over a field F.

a) Given an isomorphism $\phi : V \to V^*$ of F-vector spaces, define $B : V \times V \to F$ by $B(v,w) = [\phi(v)](w)$. Show that B is a non-degenerate bilinear pairing on V.

b) Conversely, show that every non-degenerate bilinear pairing $V \times V \to F$ arises in this way from an isomorphism $\phi : V \to V^*$. In particular, if $F = \mathbb{R}$ then every inner product on V arises from such an isomorphism.

c) If $F = \mathbb{R}$ and B is the dot product on $V = \mathbb{R}^n$, find the isomorphism $V \to V^*$ that induces B.

3. If $C \in M_n(F)$ is a symmetric matrix, define $B : F^n \times F^n \to F$ by $B(v,w) = v^t C w$, where $v, w \in F^n$ are viewed as column vectors and v^t is the transpose of v.

a) Show that B is a symmetric bilinear form. What is B if C is the identity?

b) Show that B is non-degenerate if and only if C is invertible.

4. Let V be a finite dimensional F-vector space, say with basis $\{e_1, \ldots, e_n\}$, and associated dual basis $\{x_1, \ldots, x_n\}$ on V^*. A map $q : V \to F$ is called a *quadratic form* on V if it is given by a homogeneous polynomial $\sum_{i \le j} d_{i,j} x_i x_j$ in $x_1, \ldots, x_n$. That is, $q(\sum a_i e_i) = \sum_{i \le j} d_{i,j} a_i a_j$.

a) Show that the quadratic forms on V form a vector space $\mathrm{QF}(V)$, and that each quadratic form q has the property that $q(cv) = c^2 q(v)$ for all $c \in F$ and $v \in V$.

b) If B is a symmetric bilinear form on V, define $q_B(v) = B(v,v)$. Show that the map $B \mapsto q_B$ defines a vector space homomorphism $\alpha : \mathrm{SBilin}(V) \to \mathrm{QF}(V)$, where $\mathrm{SBilin}(V)$ is the vector space of symmetric bilinear forms on V. [Hint: To show that $\alpha(B) \in \mathrm{QF}(V)$, let $C = (c_{i,j})$ be the symmetric matrix associated to B with respect to the given basis. That is, $B(v,w) = v^t C w$, where on the right hand side we write v, w as column vectors in terms of the basis $\{e_1, \ldots, e_n\}$. Now evaluate $B(v,v)$ and see that it has the right form.]

c) If B is the dot product on $\mathbb{R}^n$, find $\alpha(B) \in \mathrm{QF}(\mathbb{R}^n)$.

5. a) In the notation of problem 4(b), show that if $q = \alpha(B)$ then $q(v+w) - q(v) - q(w) = 2B(v,w)$ for all $v, w \in V$.

b) If $\mathrm{char}(F) \neq 2$, deduce that α is an isomorphism, and find the dimensions of $\mathrm{SBilin}(V)$ and $\mathrm{QF}(V)$ in terms of $\dim(V)$. What goes wrong if $\mathrm{char}(F) = 2$?

c) Let $V = F^2$ with $\mathrm{char}(F) \neq 2$. For $v = (a,b) \in V$ let $q_1(v) = a^2 - b^2$ and let $q_2(v) = a^2 + ab + b^2$. Explain why q_1, q_2 are quadratic forms on V, and find the symmetric bilinear forms B_1, B_2 on V such that $\alpha(B_i) = q_i$ for $i = 1, 2$. Also find the symmetric matrices C_1, C_2 that induce B_1, B_2 as in problem 3. What goes wrong if $\mathrm{char}(F) = 2$?