Math 502 Problem Set #9 Due week of Oct. 30, 2017, in lab.

Review Artin, Chapter 3, sections 3.3, 3.4, 3.6, 3.7; and Chapter 4, sections 4.1-4.3.

1. a) Let $V, W, X$ be finite dimensional vector spaces, and let $S : W \to X$ and $T : V \to W$ be linear transformations. Show that $\text{rank}(S \circ T) \le \text{rank}(T)$ and that $\text{rank}(S \circ T) \le \text{rank}(S)$.

b) Let $T : V \to V$ be a linear operator on a finite dimensional vector space $V$. Let $r_n$ be the rank of $T^n = T \circ T \circ \cdots \circ T$ (with $n$ copies of $T$). Prove that $r_{n+1} \le r_n$ for all $n$, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant.

c) What happens in (b) if instead $V$ is infinite dimensional?

2. Determine whether each of the following assertions is true or false. Explain.

a) If $V$ is a finite dimensional vector space with basis $B = \{v_1, \ldots, v_n\}$, and $W$ is a subspace of $V$, then $B \cap W$ is a basis for $W$.

b) If $V$ is a vector space, and $S$ is a linearly independent subset of $V$ that is not contained in *any* strictly larger linearly independent subset of $V$, then $S$ is a basis of $V$.

3. Let $V$ be an $n$-dimensional $F$-vector space, with dual space $V^* = \text{Hom}(V, F)$.

a) Let $B = \{v_1, \ldots, v_n\}$ be a basis of $V$. Show that for every $i = 1, \ldots, n$ there is a unique $\delta_i \in V^*$ such that $\delta_i(v_j) = 1$ if $i = j$ and $\delta_i(v_j) = 0$ if $i \ne j$.

b) Show that the elements $\delta_1, \ldots, \delta_n$ in part (a) form a basis $B^*$ of $V^*$.

4. a) In the notation of problem 3, show that there is a unique isomorphism $\phi_{V,B} : V \to V^*$ such that $\phi_{V,B}(v_i) = \delta_i$ for $i = 1, \ldots, n$.

b) By giving an example, show that the map $\phi_{V,B}$ can depend on the choice of the basis $B$. [Hint: Take $V = \mathbb{R}^2$, take $B = \{e_1, e_2\}$, and take $B'$ to be some other basis. Then compute both $\phi_{V,B}$ and $\phi_{V,B'}$ in terms of the bases $B, B^*$ on $V, V^*$.]

5. In the notation of problem 4 above, let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B} : V \to V^{**}$.

a) Show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism.

b) For $v \in V$, show that $\psi_{V,B}(v)$ is the element of $V^{**}$ taking $f \in V^*$ to $f(v)$.

c) Deduce that the isomorphism $\psi_{V,B} : V \to V^{**}$ is independent of the choice of $B$. Contrast this with the situation in problem 4(b). (As discussed in class, we say that $\psi_{V,B}$, unlike $\phi_{V,B}$, is a *natural* isomorphism.)