Math 502 Problem Set #7 Due week of Oct. 16, 2017, in lab.

Read Artin, Chapter 6, sections 6.1 and 6.7.

1. a) From Artin, Chapter 3, do these problems from Section 3.2 on fields: 1, 2, 5, 11. *Note:* The problems in Chapter 3 are on pages 98-101. Due to a misprint, the sections are numbered incorrectly there. The problems for section 2 of fields are on pages 98-99, listed as "Section 1 Fields".

b) From Artin, Chapter 6, do these problems (pages 188-194): 7.1, 7.7. (Note: The term "group operation" in Section 6.7 is equivalent to group action.)

2. a) Prove that for every prime p, there is exactly one field of order p up to isomorphism. This field is denoted by $\mathbb{F}_p$.

b) Show that every field F of characteristic p contains $\mathbb{F}_p$ as a subfield, and that moreover F is a vector space over $\mathbb{F}_p$.

c) Prove that if F is a finite field then $|F| = p^n$ for some prime p and some positive integer n. [Hint: What is the characteristic of F? Now use part (b). What is the dimension of this vector space?]

3. Explicitly find a field of four elements $\{0, 1, a, b\}$, by writing down the addition and multiplication tables.

4. Let G act on a set X, and let $x, x' \in X$.

a) Show that if x, x' are in the same orbit, then their stabilizers are conjugate subgroups of G. Show by example that these stabilizers need not be equal.

b) Show by example that if x, x' are *not* in the same orbit, then their stabilizers need not be conjugate.

5. a) Show that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \approx (\mathbb{Z}/n\mathbb{Z})^\times$, for any positive integer n. [The left hand side refers to automorphisms as a group.]

b) Let $G_1 = \mathbb{Z}/3\mathbb{Z}$, and for $i \geq 1$ let $G_{i+1} = \mathrm{Aut}(G_i)$. For every positive integer n find G_n, and determine which of these are abelian.

c) Do the same with $G_1 = \mathbb{Z}/8\mathbb{Z}$.