Math 371 Problem Set #8 Due week of March 17, 2008, in lab.

Read Herstein, Chapter 5, section 3.

1. From Herstein, do these problems:

a) Chapter 3, supplementary problems, pages 167-168: #7, 9, 16.
b) Chapter 5, section 1, page 215: #5, 10, 11.
c) Chapter 5, section 3, pages 227-228: #6(a,c), 7, 10(a,b). [Hint for #7: First use Herstein, §3.10, #3.]

2. Which of the following groups are cyclic? $(\mathbb{Z}/9)^\times$, $(\mathbb{Z}/10)^\times$, $(\mathbb{Z}/15)^\times$. For each one that is, find a generator.

3. a) Show that $\alpha = \cos(\pi/9)$ is an algebraic number. [Hint: Let $\beta = 2\alpha$ and evaluate $\beta^3 - 3\beta$, using the identity $2\cos(2\pi/n) = e^{2\pi i/n} + e^{-2\pi i/n}$.]
b) Do the same with α replaced by $\cos(2\pi/7)$. [Hint: Now use $\beta^3 + \beta^2 - 2\beta$.]

4. a) Show that if $\alpha \in \mathbb{Q}$ is a root of a polynomial $\sum_{i=0}^n c_i x^i \in \mathbb{Z}[x]$ with $c_0, c_n \neq 0$, and $\alpha = a/b$ in lowest terms (i.e. $a, b \in \mathbb{Z}$ are relatively prime), then $a|c_0$ and $b|c_n$.
b) Explain why this generalizes problem 11 in §5.1 of Herstein.
c) Show that $\sqrt[3]{2}$, $\cos(\pi/9)$, and $\cos(2\pi/7)$ are each algebraic of degree 3 over $\mathbb{Q}$; i.e., in each case, 3 is the degree of the minimal polynomial satisfied by the number. [Hint: Apply part (a), and use problem 3 above.]