Math 371 Problem Set #5 Due week of Feb. 18, 2008, in lab.

Reminder: Exam #1 will take place in class on Wednesday, February 20, on the material covered in class up through Friday, February 15, and the related material on homeworks.

Read Herstein, Chapter 3, sections 5-7 and 9.

1. From Herstein, Chapter 3, do these problems:

a) Section 3.4, page 136: #14, 17.

b) Section 3.5, page 139: #1.

c) Section 3.6, page 142: #4.

d) Section 3.7, page 149: #2, 7.

e) Section 3.9, page 158: #1(a), #2(b,d).

2. Show that the matrix ring $M_2(\mathbb{R})$ has no (two-sided) proper ideals, yet is not a division ring. Could this happen for a commutative ring?

3. a) Show that if $a, b \in \mathbb{R}$ and $a \neq 0$, then the ideal $(ax + b)$ is a maximal ideal of $\mathbb{R}[x]$, and that $\mathbb{R}[x]/(ax + b) \approx \mathbb{R}$.

b) Show that if $a, b, c \in \mathbb{R}$ and $a \neq 0$, then the ideal $(ax^2 + bx + c)$ is a maximal ideal of $\mathbb{R}[x]$ if and only if $b^2 - 4ac < 0$. Under what circumstances is $\mathbb{R}[x]/(ax^2 + bx + c)$ a field, and what field is it isomorphic to?

c) Show that if $a, b, c, d \in \mathbb{R}$ and $a \neq 0$, then the ideal $(ax^3 + bx^2 + cx + d)$ is not a maximal ideal of $\mathbb{R}[x]$. Can $\mathbb{R}[x]/(ax^3 + bx^2 + cx + d)$ be a field?

4. a) Find an irreducible polynomial $f(x) \in \mathbb{F}_2[x]$ of degree 2, and show that $\mathbb{F}_2[x]/(f(x))$ is a field. How many elements does it have?

b) Do the same with an irreducible polynomial in $\mathbb{F}_2[x]$ of degree 3.

5. Find the fraction field (i.e. field of quotients) for each of the following: $\mathbb{Z}[1/2]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}_{(2)} := \{a/b \mid a, b \in \mathbb{Z},\ 2 \nmid b\}$, $\mathbb{Z}[i, x]$.