Math 371 Problem Set #4 Due week of Feb. 11, 2008, in lab.

Read Herstein, Chapter 3, sections 3-4.

1. From Herstein, Chapter 3, do these problems:

a) Section 3.2, page 130: #6, 7, 13 (note that #13 asks for a *new* proof of p.24, #15).

b) Section 3.4, page 135: #3, 5-8.

2. Which of the following are ring homomorphisms? For those that are not, why not? For those that are, find the kernel and image.

i) $\mathbb{R}[x] \to \mathbb{C}$, $f(x) \mapsto f(3)$.
ii) $\mathbb{R}[x] \to \mathbb{C}$, $f(x) \mapsto f(2i)$.
iii) $\mathbb{C} \to \mathbb{R}$, $a + bi \mapsto a$ for $a, b \in \mathbb{R}$.
iv) $\mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$, $a + b\sqrt{2} \mapsto a + b\sqrt{3}$ for $a, b \in \mathbb{Q}$.
v) $\mathbb{Q}[\zeta] \to \mathbb{Q}[\zeta]$, $a + b\zeta \mapsto a + b\zeta^2$, for $a, b \in \mathbb{Q}$, where $\zeta = e^{2\pi i/3}$.
vi) $\mathbb{Z}[i] \to \mathbb{Z}/5$, $a + bi \mapsto a + 2b$, for $a, b \in \mathbb{Z}$.
vii) $\mathbb{C} \to M_2(\mathbb{R})$, $a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for $a, b \in \mathbb{R}$.

3. Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is a homomorphism of rings.
a) Show that $\phi(r) = r$ for all $r \in \mathbb{Z}$.
b) Do the same for all $r \in \mathbb{Q}$.
c) Show that if $r \geq 0$ then $\phi(r) \geq 0$. [Hint: $r \geq 0 \Leftrightarrow r = s^2$ for some s.]
d) Show that ϕ is an increasing function. [Hint: Part (c).]
e) Conclude that ϕ is the identity. [Hint: Parts (b) and (d).]

4. Let $\mathbb{H}$ be the ring of quaternions $\alpha = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$. Define the conjugate $\bar{\alpha} = a - bi - cj - dk$, and the absolute value $|\alpha| \geq 0$ by $|\alpha|^2 = a^2 + b^2 + c^2 + d^2$.

a) Show that $|\alpha|^2 = \alpha\bar{\alpha}$ and that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$. Conclude that $|\alpha\beta| = |\alpha||\beta|$. Also, find all $\alpha \in \mathbb{H}$ such that $|\alpha| = 0$.

b) Show that $\mathbb{H}$ does not have any zero-divisors. [Hint: Use part (a).]