Math 370 Sample Final Exam May 2014

Instructions: This exam consists of ten problems. Do all ten, showing your work and explaining your assertions. Allow yourself two hours. Each problem is worth 10 points, for a total of 100 points.

1. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & c & 0 \end{pmatrix} \in M_3(\mathbb{R})$ where $c \in \mathbb{R}$ is a scalar.

a) Using determinants, find all values of c for which A is invertible.

b) Find all values of c for which there is a *unique* solution to this system of equations:

$$\begin{array}{rrrl} x & +2y & -z & = 17 \\ & y & +2z & = 18 \\ x & +cy & & = 19 \end{array}$$

2. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ whose kernel consists of the multiples of the vector $(1, 2, 3)$. Find the image of T.

3. Let V be the vector space of continuous functions $\mathbb{R} \to \mathbb{R}$. Define $T : V \to V$ by $T(f) = f \circ \sin$. Determine whether T is a linear transformation, whether it is injective, whether it is surjective, and whether it is an isomorphism.

4. Find the monic polynomial of least degree in the ideal of $\mathbb{R}[x]$ generated by $x^3 + x$ and $x^4 - 1$.

5. Define $T : \mathbb{R}^3 \to \mathbb{R}^4$ by $T(a, b, c) = (a - b, b - c, c - a, 2a - b - c)$. Find the rank and the nullity of T. Do the same for the induced linear transformation $T^{\mathrm{t}} : (\mathbb{R}^4)^* \to (\mathbb{R}^3)^*$ on the dual spaces.

6. Prove that the real matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ are not similar.

7. Determine whether the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ is diagonalizable. If it is, find a diagonal matrix that is similar to A. Justify your assertions.

8. For $v = (a, b)$ and $w = (c, d)$ in $\mathbb{R}^2$, define $\langle v, w \rangle = ac - ad - bc + 2bd$. Show that this is an inner product, and find an orthonormal basis of $\mathbb{R}^2$ with respect to this inner product.

9. Find real numbers a, b such that with respect to the usual inner product (dot product) on $\mathbb{R}^3$, there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of $\begin{pmatrix} 1 & 0 & a \\ 0 & 2 & b \\ -1 & 3 & 0 \end{pmatrix}$.

10. a) Find the eigenvalues of $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

b) Show that there is a real matrix B such that $B^5 = A$.