Math 370 Problem Set #13 Due week of April 21, 2014, in lab.

Read Hoffman and Kunze, Chapter 6, Sections 1-3.

1. From Hoffman and Kunze, Chapter 6, do these problems: Page 189, #1,3,5.

2. Let $c_1, \ldots, c_k \in F$ be distinct elements in a field F.

a) Show that there is no non-zero vector $(b_0, \ldots, b_{k-1}) \in F^k$ such that $\sum_{j=0}^{k-1} b_j c_i^j = 0$ for all $i = 1, \ldots, k$. [Hint: Consider roots of the polynomial $f(x) = \sum_{j=0}^{k-1} b_j x^j$.]

b) Let $a_1, \ldots, a_k \in F$. Using part (a), show that the matrix

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ a_1c_1 & a_2c_2 & \ldots & a_kc_k \\ a_1c_1^2 & a_2c_2^2 & \ldots & a_kc_k^2 \\ \vdots & \vdots & & \vdots \\ a_1c_1^{k-1} & a_2c_2^{k-1} & \ldots & a_kc_k^{k-1} \end{pmatrix}$$

is invertible, unless some $a_i = 0$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ take $(a, b) \mapsto (17a - 30b, 9a - 16b)$.

a) Find the matrix of T with respect to the standard basis e_1, e_2.

b) Find the matrix of T with respect to the basis $f_1 = (1, 1)$, $f_2 = (1, -1)$.

c) Find a basis of $\mathbb{R}^2$ for which the matrix of T is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

d) Is there a basis of $\mathbb{R}^2$ for which T has matrix $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$? Explain.

4. Let $T : V \to V$ be a linear transformation, and let S be the set of eigenvectors of T (including 0).

a) Show that if c is an eigenvalue of T, then the eigenspace $W_c = \{v \in V \mid T(v) = cv\}$ is really a subspace of V, but that S is *not* always a subspace of V. Can S *ever* be a subspace of V?

b) Let W be the span of S. Show that W is *invariant* under T, i.e. $T(W) \subset W$. Must $W = V$?

5. Let $\mathcal{P}_2$ be the real vector space of polynomials in $\mathbb{R}[x]$ of degree at most 2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by $T(f) = g$ where $g(x) = (x + 1)f'(x)$. Find the eigenvalues and eigenvectors of T. Do this in *two different* ways, namely:

i) Find the matrix of T relative to the basis $\{1, x, x^2\}$ and use that.

ii) Instead, use separation of variables to solve the differential equation $(x+1) \cdot \frac{dy}{dx} = cy$, where c is a constant.

6. a) Show that if $T : V \to V$ is a linear transformation, and $v \in V$ is an eigenvector for T with eigenvalue c, then v is also an eigenvector for T^k, with eigenvalue c^k.

b) Use this to find the eigenvalues and corresponding eigenvectors of A^{253}, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

[Note: You are not required to find the entries of the matrix A^{253}.]