Math 370 Problem Set #7 Due week of March 3, 2014, in lab.

Read Hoffman and Kunze, Chapter 3, Sections 4 and 5.

1. From Hoffman and Kunze, Chapter 3, do these problems:
Pages 95-96, #1(a,b), 5, 8. Page 105, #2, 3, 5.

2. Suppose that A and B are similar $n \times n$ matrices (i.e. $B = C^{-1}AC$ for some invertible $n \times n$ matrix C).

a) Show that A^n and B^n are similar.
b) Show that if $f(x)$ is a polynomial, then $f(A)$ and $f(B)$ are similar.
c) Show that if $f(x)$ is a polynomial such that $f(A) = 0$, then $f(B) = 0$.

3. Let $T : V \to V$ be a linear transformation, where V is a finite dimensional vector space. Let T^n denote $T \circ T \circ \cdots \circ T$ (with n factors of T) and let $r_n = \text{rank}(T_n)$. Prove that $r_{n+1} \le r_n$ for all n, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant. (Hint: See Problem Set 6, #2.)

4. Let $\mathcal{P}_n$ be the vector space of real polynomials $f(x)$ of degree at most n. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $f(x) \mapsto f(x) - (x-1)f'(x)$, where $f'(x)$ is the derivative of $f(x)$.

a) Show that T is a linear transformation.
b) Find the kernel of T. (Hint: Differential equations.)
c) Find the matrix of T with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$.
d) Find the matrix of T with respect to the basis $\{f_1, f_2, f_3\}$ of $\mathcal{P}_2$, where $f_i = F^{-1}(e_i)$ as in problem 6 of Problem Set #6.

5. a) Which of the following are groups?

i) {non-zero rational numbers}, under multiplication.
ii) {2×2 real matrices having positive determinant}, under matrix multiplication.
iii) {2×2 real matrices having positive trace}, under matrix addition.

For each one that is, determine whether it is commutative.

b) Which of the following maps $\text{GL}_2(\mathbb{R}) \to \text{GL}_2(\mathbb{R})$ are group homomorphisms?

i) $A \mapsto A^2$.
ii) $A \mapsto A^{-1}$.
iii) $A \mapsto C^{-1}AC$, where C is a given 2×2 invertible matrix.
iv) $A \mapsto \begin{pmatrix} \det(A^2) & 0 \\ 0 & 1 \end{pmatrix}$.

For each one that is, find the kernel and image.