Math 370 Problem Set #6 Due week of Feb. 24, 2014, in lab.

Read Hoffman and Kunze, Chapter 3, Sections 2 and 3.

1. From Hoffman and Kunze, Chapter 3, do these problems:
Page 83, #3. Pages 85-86, #1,6,7.

2. Let $T : V \to W$ and $S : W \to X$ be linear transformations of vector spaces. What inequalities (i.e. $\leq$) can you find relating these quantities: $\operatorname{rank}(T)$, $\operatorname{rank}(S)$, $\operatorname{rank}(S \circ T)$, $\dim(V)$, $\dim(W)$, $\dim(X)$? (For example, can you relate $\operatorname{rank}(S \circ T)$ to all the other quantities?)

3. Let A be a 2×3 matrix. Show that there cannot be any 3×2 matrix B such that BA is the 3×3 identity matrix. But also show that (depending on A) it may be possible to find a 3×2 matrix B such that AB is the 2×2 identity matrix. (Hint: Problem 2 may help.)

4. Let $W \subset \mathbb{R}^3$ be the subspace given by $x + y + z = 0$. Find a basis of W and extend it to a basis of $\mathbb{R}^3$. Find another basis of $\mathbb{R}^3$ that does not contain *any* vector in W.

5. Let $T : V \to V$ be a linear transformation. Show that $\operatorname{image}(T) \subset \operatorname{kernel}(T) \Leftrightarrow T^2 = 0$.

6. Let n be a non-negative integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $\mathcal{P}_n$ be the vector space of real polynomials $f(x)$ of degree $\leq n$. Define $F : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.

a) Show that F is an isomorphism. [Hint: $\dim(\mathcal{P}_n) =$? $\operatorname{kernel}(F) =$?]

b) *Explicitly* find $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ (where $e_1, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{R}^{n+1}$) in the case $n = 2$ and $\alpha_j = j$ for $j = 1, 2, 3$. [Hint: Where does $(x - a)(x - b)$ vanish?]

c) In general, deduce that $F^{-1}(e_1), \ldots, F^{-1}(e_{n+1})$ form a basis of $\mathcal{P}_n$. In the special case done in (b), express the polynomial x as a linear combination of these basis elements.