Math 314 Problem Set #8 Due week of March 18, 2019, in lab.

In Hoffman and Kunze, read Chapter 3, Section 7; and Chapter 4 (all).

1. From Hoffman and Kunze, Chapter 3, do these problems: pages 105-107, #11, 12, 17; pages 115-116, #1, 7. From Chapter 4, do these problems: pages 122-123, #1(a), 4, 6, 7.

2. a) Show that if $T : V \to W$ is a surjective linear transformation of finite dimensional vector spaces with kernel N, then $\dim(V/N) = \dim(V) - \dim(N)$. [Hint: Consider $\dim(W)$.]

b) Illustrate this with the example $V = \mathbb{R}^3$, $W = \mathbb{R}$, $T(x, y, z) = x + y + z$.

3. Let $T : V \to W$ and $S : W \to Z$ be linear transformations of finite dimensional vector spaces. Consider the transpose transformations, $T^* : W^* \to V^*$ and $S^* : Z^* \to W^*$.

a) Show that $T^* = 0$ if and only if $T = 0$.

b) Show that $(S \circ T)^* = T^* \circ S^*$, and deduce that if $S \circ T = 0$ then $T^* \circ S^* = 0$. [You can do this either using the linear transformations or the corresponding matrices.]

c) Show that if T is surjective then T^* is injective. [Hint: What is the kernel of T^*?]

d) Show that if T is injective then T^* is surjective. [Hint: Pick a basis $\mathcal{B}$ of V, and show that $T(\mathcal{B}) \subset W$ extends to a basis of W.]

e) Conclude that T is an isomorphism if and only if T^* is an isomorphism.

4. Let $\mathcal{A}$ be an algebra over a field F.

a) Show that $0 \cdot a = 0$ for all $a \in \mathcal{A}$, where $0 \in \mathcal{A}$ is the additive identity in $\mathcal{A}$.

b) Suppose that $\mathcal{A}$ has a multiplicative identity $1 \in \mathcal{A}$. Prove that $(-1) \cdot a = -a$ for all $a \in \mathcal{A}$ (where $-a \in \mathcal{A}$ denotes the additive inverse of $a \in \mathcal{A}$).

5. a) Let F be a field, let $f(x) \in F[x]$, and let A be an $n \times n$ matrix over F. Suppose that $f(x) = f_1(x)f_2(x)$ in $F[x]$. Prove that $f(A) = f_1(A)f_2(A)$ as matrices.

b) Let $f(x, y) \in F[x, y]$, the algebra of polynomials in x and y with coefficients in F. Let A, B be $n \times n$ matrices over F. Suppose that $f(x, y) = f_1(x, y)f_2(x, y)$ in $F[x, y]$. Show that $f(A, B)$ is *not* necessarily equal to the matrix $f_1(A, B)f_2(A, B)$. [Hint: Let $f(x, y) = x^2 - y^2$ and pick two 2×2 matrices.]

c) Explain where your proof for (a) breaks down in (b).